Homework
due Wednesday, December 4

Read carefully sections 39, 40 in Chapter 7 of Hartshorne’s book. Solve the following problems.

**Problem 1.** Let $\Pi$ be a Hilbert plane. For any line $l$ in $\Pi$ the reflection in $l$ is the function $s_l : \Pi \to \Pi$ defined as follows: for a point $A$ in $\Pi$ there is unique line $t$ through $A$ perpendicular to $l$. Let $P$ be the point where $l$ and $t$ intersect. If $A = P$ then define $s_l(A) = A$. If $A \neq P$ then there is unique point $B$ on $t$ such that $A \ast P \ast B$ and $\overline{AP} \equiv \overline{PB}$. Define $s_l(A) = B$.

A **rigid motion** is a function $R : \Pi \to \Pi$ such that

1. if $l$ is a line then its image $R(l) = \{R(A) : A \in l\}$ is also a line.
2. if $A \ast B \ast C$ then $R(A) \ast R(B) \ast R(C)$.
3. $\overline{AB} \equiv R(\overline{A})R(\overline{B})$ for every segment $\overline{AB}$.
4. $\angle ABC \equiv \angle R(\overline{A})R(\overline{B})R(\overline{C})$ for every angle $\angle ABC$.

a) Prove that any rigid motion is a bijection and the inverse function is also a rigid motion.

b) Prove that a composition of rigid motions is a rigid motion.

c) Prove that any reflection in a line is a rigid motion.

d) Prove that if a rigid motion fixes three non-collinear points then it is the identity function (i.e. it fixes all the points).

e) Prove that any rigid motion $R$ is a composition of at most three reflections. Hint: Pick three non-collinear points $A, B, C$. Find a composition $S$ of at most three reflections such that $R(A) = S(A)$, $R(B) = S(B)$, $R(C) = S(C)$. Conclude that $S = R$.

Problem 39.3
Problem 39.8
Problem 39.9
Problem 39.14 part b)
Problem 39.19

Problem 39.11 Hint: We can find the Euclidean center $C$ of $\zeta$ (how?). Consider the line $OC$. Explain why the P-center of $\zeta$ is on $OC$. Now we may proceed in various ways.

**First approach:** It suffices to construct a circle $T$ which is perpendicular to both $\Gamma$ and $\zeta$. Explain why $T$ will give a $P$-line which contains a $P$-diameter of $\zeta$ (recall: in any Hilbert plane a tangent line to a circle at some point is perpendicular to the diameter from that point). Thus the point inside $\zeta$ where $T$ intersects the line $OC$ is the $P$-center of $\zeta$. To construct $T$, note first that it is easy to construct a circle perpendicular to a given line and a given circle. Then use appropriate inversion to reduce to this case.

**Second approach:** Use first appropriate inversion to reduce to the case when $\zeta$ passes through $O$. Assuming that $\zeta$ contains $O$, the idea from the first approach becomes even simpler. Let $A$ be the second point of intersection of the line $OC$ and $\zeta$. Find the inverse $B$ of $A$ in the circle $\Gamma$. Then $B$ is outside of $\Gamma$. Draw a tangent to $\Gamma$ from $B$ and let $D$ be the point of tangency. Prove that the circle with center at $B$ and radius $BD$ intersects the segment $OA$ at the $P$-center of $\zeta$.

Regardless of which approach you use, carefully explain how you do the appropriate inversions using a compas and a ruler.