Homework

due Wednesday, December 4

Read carefully sections 39, 40 in Chapter 7 of Hartshorne's book. Solve the following problems.

Problem 1. Let Π be a Hilbert plane. For any line l in Π the **reflection in** l is the function $s_l : \Pi \longrightarrow \Pi$ defined as follows: for a point A in Π there is unique line t through A perpendicular to l. Let P be the point where l and t intersect. If $A = P$ then define $s_l(A) = A$. If $A \neq P$ then there is unique point B on t such that $A * P * B$ and $\overline{AP} \equiv \overline{PB}$. Define $s_l(A) = B$.

A **rigid motion** is a function $R : \Pi \longrightarrow \Pi$ such that

1. if l is a line then its image $R(l) = \{R(A) : A \in l\}$ is also a line.
2. if $A * B * C$ then $R(A) * R(B) * R(C)$.
3. $\overline{AB} \equiv \overline{R(A)R(B)}$ for every segment $\overline{AB}$.
4. $\angle ABC \equiv \angle R(A)R(B)R(C)$ for every angle $\angle ABC$.

a) Prove that any rigid motion is a bijection and the inverse function is also a rigid motion.

b) Prove that a composition of rigid motions is a rigid motion.

c) Prove that any reflection in a line is a rigid motion.

d) Prove that if a rigid motion fixes three non-collinear points then it is the identity function (i.e. it fixes all the points).

e) Prove that any rigid motion R is a composition of at most three reflections. Hint: Pick three non-collinear points A, B, C. Find a composition S of at most three reflections such that $R(A) = S(A)$, $R(B) = S(B)$, $R(C) = S(C)$. Conclude that $S = R$.

Problem 39.3

Problem 39.8

Problem 39.9

Problem 39.14 part b)

Problem 39.19

Problem 39.11 Hint: We can find the Euclidean center C of ζ (how ?). Consider the line OC. Explain why the P-center of ζ is on OC. Now we may proceed in various ways.

First approach: It suffices to construct a circle T which is perpendicular to both Γ and ζ. Explain why T will give a P-line which contains a P-diameter of ζ (recall: in any Hilbert plane a tangent line to a circle at some point is perpendicular to the diameter from that point). Thus the point inside ζ where T intersects the line OC is the P-center of ζ. To construct T, note first that it is easy to construct a circle perpendicular to a given line and a given circle. Then use appropriate inversion to reduce to this case.

Second approach: Use first appropriate inversion to reduce to the case when ζ passes through O. Assuming that ζ contains O, the idea from the first approach becomes even simpler. Let A be the second point of intersection of the line OC and ζ. Find the inverse B of A in the circle Γ. Then B is outside of Γ. Draw a tangent to Γ from B and let D be the point of tangency. Prove that the circle with center at B and radius BD intersects the segment OA at the P-center of ζ.

Regardless of which approach you use, carefully explain how you do the appropriate inversions using a compas and a ruler.