Homework 3

Problem 1: Six points are positioned in a 3×4 rectangle. Prove that there are two of them at the distance not exceeding $\sqrt{5}$ from each other.

Problem 2: Let $-1 \le a < b \le 1$. Prove that there is a natural number n such that $a < \sin n < b$ (you may use the fact that π is irrational).

Problem 3: Nine distinct points with all coordinates integral are selected in the space. Prove that the line segment with ends at certain two of these points contains in its interior a point with all coordinates integral.

Problem 4: 21 integers are selected from $\{1, 2, 3, ..., 400\}$. Prove that two of them, say x and y, satisfy $0 < |\sqrt{x} - \sqrt{y}| < 1$.

Problem 5: 2004 flies are inside a cube of side 1. Show that some 3 of them are within a sphere of radius $1/11$.

Problem 6: Consider a system of equations with integer coefficients a_i, $1 \le i \le 6$, with all $|a_i| \le 10$:

$$a_1 x + a_2 y + a_3 z = 0, \quad a_4 x + a_5 y + a_6 z = 0.$$

Prove that this system has a solution in integers x, y, z which are not all 0 and satisfy $|x| \le 900$, $|y| \le 900$ and $|z| \le 900$.