Homework 5

Problem 1: Find the last digit of $\left[\dfrac{10^{20000}}{10^{100}+3}\right]$.

Problem 2: Problem 5.1.10

Problem 3: Problem 5.1.13 (Hint: Given n, when is $\left[\dfrac{n+2^k}{2^{k+1}}\right] \neq \left[\dfrac{n+1+2^k}{2^{k+1}}\right]$?)

Problem 4: Let p be an odd prime.

1) Show that there is a prime $q \not\equiv 1 \pmod{p^2}$ such that $q|(1+p+p^2+...p^{p-1})$.

2) Show that if q is as in 1) then $n^p \not\equiv p$ (mod q) for all integers n.

Problem 5: Show that there exists a natural number n which has exactly 2004 prime divisors and such that $n|(2^n+1)$. Hint: Show first that $3^k|(2^{3^k}+1)$ for every k.

Problem 6: 1) Prove that if a prime p divides 2^n-1 then n has a prime divisor smaller than p.

2) Suppose that natural numbers n_1, n_2,...,n_k satisfy

$$n_1|2^{n_2}-1,\ n_2|2^{n_3}-1,\ ...,\ n_{k-1}|2^{n_k}-1,\ n_k|2^{n_1}-1.$$

Prove that $n_1 = n_2 = ... = n_k = 1$.

Problem 7: Let p be an odd prime number. For $0 < k < p$ denote by h_k the unique integer such that $0 < h_k < p$ and $kh_k \equiv 1 \pmod{p}$.

1) Prove that $p|(h_1 - h_2 + h_3 - ... \pm h_m)$, where $m = [2p/3]$.

2) Use 1) to prove that $\binom{p}{1}+\binom{p}{2}+...+\binom{p}{m}$ is divisible by p^2, where $m = [2p/3]$. Hint: Show that $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$.

3) Prove that $\dfrac{2^p-2}{p} \equiv h_{k+1} + h_{k+1} + ... + h_{2k} \pmod{p}$, where $p = 2k+1$. Hint: Note that $2^p = \binom{p}{0} + \binom{p}{1} + ... + \binom{p}{p}$.