Homework 3

Problem 1: Let p be an odd prime number. For $0 < k < p$ denote by h_k the unique integer such that $0 < h_k < p$ and $kh_k \equiv 1 \pmod{p}$.

1) Prove that $h_i + h_{p-i} \equiv 0 \pmod{p}$ for $i = 1, 2, ..., p-1$.

2) Prove that $h_i \equiv 2h_{2i} \pmod{p}$ for $i = 1, 2, ..., (p-1)/2$.

3) Prove that $h_1 - h_2 + h_3 - ... \pm h_m \equiv 0 \pmod{p}$, where $m = \lfloor 2p/3 \rfloor$.

4) Prove that $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ and $\frac{1}{p}\binom{p}{k} \equiv (-1)^{k-1} h_k \pmod{p}$ for $k = 1, 2, ..., p-1$.

5) Use 3) and 4) to prove that $\binom{p}{1} + \binom{p}{2} + ... + \binom{p}{m}$ is divisible by p^2, where $m = \lfloor 2p/3 \rfloor$.

6) Prove that $\dfrac{2^p - 2}{p} \equiv h_{k+1} + h_{k+2} + ... + h_{2k} \pmod{p}$, where $p = 2k+1$.
Hint: Recall that $2^n = \binom{n}{0} + \binom{n}{1} + ... + \binom{n}{n}$.

Problem 2: 1) Prove that if a prime p divides $2^n - 1$ then n has a prime divisor smaller than p.

2) Suppose that the natural numbers n_1, n_2,...,n_k satisfy

$$n_1 | 2^{n_2} - 1,\ n_2 | 2^{n_3} - 1,\ ...,\ n_{k-1} | 2^{n_k} - 1,\ n_k | 2^{n_1} - 1.$$

Prove that $n_1 = n_2 = ... = n_k = 1$.

Problem 3: Show that there exists a natural number n which has exactly 2006 prime divisors and such that $n | (2^n + 1)$. Hint: Show first that $3^k | (2^{3^k} + 1)$ for every k. Show that if k is large then $2^{3^k} + 1$ has more that 2006 prime divisors. Note that if $m | (a+1)$ and m is odd then $m | (a^m + 1)$.

Problem 4: Find the last digit of $\left\lfloor \dfrac{10^{20000}}{10^{100} + 3} \right\rfloor$.

Problem 5: Prove that the product of three consecutive natural numbers is never a square of an integer. Prove more generally, that it cannot be an m-th power of an integer for any $m > 1$.