Solutions to Exam 1

Problem 1. Let S be a set consisting of 10 natural numbers not exceeding 100. Show that there are two different subsets of S such that the sums of the numbers in each subset are equal. (Hint. Recall that a set with n elements has $2^n - 1$ non-empty subsets)

Solution: For every non empty subset T of S let σ_T be the sum of all numbers in T. Note that $\sigma_T \leq 10 \cdot 100 = 1000$. Thus we assign to each of the $2^{10} - 1 = 1023$ nonempty subsets of S one of the 1000 integers between 1 and 1000. By the pigeonhole principle (subsets of S are the pigeons; natural numbers not exceeding 1000 are the holes) two different subsets must be assigned the same number.

Problem 2. Numbers $1, 2, 3, ..., 2006$ are written on a blackboard. Every now and then somebody picks two numbers a and b and replaces them by $a - 1$, $b + 3$. Is it possible that at some point all numbers on the blackboard are even? Can they all be odd?

Solution: Let us look at the parity of the sum of all the numbers on the blackboard. Each time somebody changes the numbers, the sum of all the numbers increases by $2 = (a - 1 + b + 3) - (a + b)$. Thus the parity of the sum of all the numbers does not change. Originally, the sum of all the numbers $1+2+...+2006 = 2006 \cdot 2007/2 = 1003 \cdot 2007$ is odd, so it is will always be odd. Since the sum of any number of even integers is even and the sum of an even number of odd numbers is even as well, we can not have all the numbers on the blackboard of the same parity.

Second method: look at the number of odd numbers on the blackboard. At the beginning we have 1003 odd numbers, i.e. an odd number of odd numbers. Each change either increases the number of odd numbers by 2 (if both a and b are even), decreases this number by two (if both a and b are odd), or leaves it unchanged (if a, b have opposite parities). Thus we have always an odd number of odd numbers on the blackboard.

Problem 3. Prove that there is no integer $n > 2$ such that $n(n + 6)$ is a square of an integer.

Solution: Note that if $n > 2$ then

$$(n+2)^2 = n^2 + 4n + 4 < n^2 + 4n + 2n = n(n+6) < n^2 + 6n + 9 = (n+3)^2.$$

Thus $n(n+6)$ is between two consecutive squares, so it can not be a square.

Problem 4. Find

$$\int_{-1}^{1} \frac{x^{2006}dx}{2^x + 1}.$$

Solution: Set $x = -y$ so

$$\int_{-1}^{1} \frac{x^{2006}dx}{2^x + 1} = \int_{1}^{-1} \frac{(-y)^{2006}(-dy)}{2^{-y} + 1} = \int_{-1}^{1} \frac{x^{2006}dx}{2^{-x} + 1}.$$

Thus

$$2\int_{-1}^{1} \frac{x^{2006}dx}{2^x + 1} = \int_{-1}^{1} \frac{x^{2006}dx}{2^x + 1} + \int_{-1}^{1} \frac{x^{2006}dx}{2^{-x} + 1} = \int_{-1}^{1} (\frac{x^{2006}}{2^x + 1} + \frac{x^{2006}}{2^{-x} + 1})dx.$$

Note that $1/(2^x + 1) + 1/(2^{-x} + 1) = 1$. Thus

$$2\int_{-1}^{1} \frac{x^{2006}dx}{2^x + 1} = \int_{-1}^{1} x^{2006}dx = \frac{2}{2007}.$$

In other words,

$$\int_{-1}^{1} \frac{x^{2006}dx}{2^x + 1} = \frac{1}{2007}.$$