Problem A1: Find the volume of the region of points (x, y, z) such that

$$(x^2 + y^2 + z^2 + 8)^2 \leq 36(x^2 + y^2).$$

Solution: Set $r^2 = x^2 + y^2$. The intersection of this region with the plane $z = t$ is the set S_t of points (x, y, t) such that

$$0 \geq r^2 - 6r + t^2 + 8 = (r - 3)^2 + t^2 - 1,$$

i.e.

$$3 - \sqrt{1 - t^2} \leq r \leq 3 + \sqrt{1 - t^2}.$$

In other words, S_t is a ring with inner circle of radius $3 - \sqrt{1 - t^2}$ and outer circle of radius $3 + \sqrt{1 - t^2}$. Thus the area of S_t is $12\pi\sqrt{1 - t^2}$. It follows that the volume in question is equal to

$$\int_{-1}^{1} S_t \, dt = \int_{-1}^{1} 12\pi\sqrt{1 - t^2} \, dt = 6\pi^2.$$

Problem A2: Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

Solution: Let us call n winning if Bob has a winning strategy. Call n loosing if Alice has a winning strategy. The key observation is that each n is either winning or loosing. In fact, 1 is loosing. If all positive integers smaller than n are either winning or loosing then n is loosing if there is a prime p such that $n + 1 - p$ is winning and n is winning otherwise.

Suppose that the set W of all winning numbers is finite so there is $k > 1$ such that every $n \geq k$ is loosing. This means that for every $n \geq k$ there is a prime p such that $n + 1 - p$ is winning. In particular, the distance of $n + 1$ to the nearest prime is at most k for every $n \geq k$. Note however that this is not possible since all the numbers $(k!)^3 \pm i$ are composite for $i = 0, 1, 2, 3, ..., k$.

Problem A3: Let $1, 2, 3, \ldots, 2005, 2006, 2007, 2009, 2012, 2016, \ldots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \ldots, 2006$ and $x_{k+1} = x_k + x_{k-2005}$ for $k \geq 2006$. Show that the sequence has 2005 consecutive terms each divisible by 2006.

Solution: For $k \leq 0$ define $x_k = x_{k+2006} - x_{k+2005}$. Then x_h is defined for every integer and satisfies the recursive relation $x_{k+1} = x_k + x_{k-2005}$. Note that $x_0 = x_{-1} = ... = x{-2004} = 1$ and $x_{-2005} = x_{-2006} = ... x_{-4009} = 0$. There are integers m and $k > 0$ such that $x_{m+i} - x_{m+i+k}$ are divisible by 2006 for $i = 0, 1, 2, ..., 2005$. It follows from the recursive formula by obvious induction that $x_{n+k} - x_n$ is divisible by 2006 for all n. Since $x_{-4009}, x_{-4008}, ..., x_{-2005}$ are all divisible by 2006, the same is true for the 2005 numbers $x_{5000k-4009}, x_{5000k-4008}, ..., x_{5000k-2005}$.

Problem A4: Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

(For example, if $n = 5$ and π takes values at $1, 2, 3, 4, 5$ of $2, 1, 4, 5, 3$, then π has a local maximum of 2 at $k = 1$, and a local maximum of 5 at $k = 4$.) What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Solution: Let $M(\pi)$ be the number of local maxima of π. For each k let p_k be the number of permutations which have a local maximum at k. The problem asks us to evaluate

$$\sum_{\pi} M(\pi)/n!.$$

Note that $\sum_{\pi} M(\pi) = \sum_{k=1}^{n} p_k$. The computation of p_k is quite easy. To get a permutation with a local maximum at 1 we need to choose two elements for $\pi(1) > \pi(2)$, which can be done in $\binom{n}{2}$ ways and then order the remaining elements in $(n-2)!$ ways, so $p_1 = \binom{n}{2}(n-2)! = n!/2$ and the same argument shows that $p_n = n!/2$. To get a permutation with a local maximum at $1 < k < n$ we need to choose three elements for $\pi(k-1) < \pi(k) > \pi(k+1)$, which can be done in $2\binom{n}{3}$ ways and then order the remaining elements in $(n-3)!$ ways. Thus $p_k = 2\binom{n}{3}(n-3)! = n!/3$. It follows that

$$\sum_{k=1}^{n} p_k = 2 \cdot \frac{n!}{2} + (n-2)\frac{n!}{3} = n!\frac{n+1}{3}.$$

The average number of local maxima is then equal to $(n+1)/3$.

Problem A5: Let n be a positive odd integer and let θ be a real number such that θ/π is irrational. Set $a_k = \tan(\theta + k\pi/n)$, $k = 1, 2, \ldots, n$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n}$$

is an integer, and determine its value.

Solution: Recall that $\tan\alpha = (e^{2i\alpha} - 1)/i(e^{2i\alpha} + 1)$. It follows that

$$a_k = \frac{e^{2i(\theta + k\pi/n)} - 1}{i(e^{2i(\theta + k\pi/n)} + 1)} = i\frac{e^{-2i\theta} - e^{2\pi ik/n}}{e^{-2i\theta} + e^{2\pi ik/n}}$$

Set $z = e^{-2i\theta}$. Recall that $\prod_{i=1}^{n}(z - e^{2\pi ik/n}) = z^n - 1$ and $\prod_{i=1}^{n}(z + e^{2\pi ik/n}) = z^n - (-1)^n$. It follows that

$$a_1 a_2 \ldots a_n = i^n \frac{z^n - 1}{z^n + 1}.$$

Now

$$a_1 + ... + a_n = i \sum_{k=1}^{n} \frac{z - e^{2\pi ik/n}}{z + e^{2\pi ik/n}}$$

Note that

$$(z^n + 1)\frac{z - e^{2\pi ik/n}}{z + e^{2\pi ik/n}}$$

is a monic polynomial of degree n which vanishes at $-e^{2\pi il/n}$ for all $1 \leq l \leq n$, $l \neq k$ and assumes value $-2n$ at $-e^{2\pi ik/n}$ (in general, if a is a root of a polynomial $f(z)$ then $g(z) = f(z)/(z-a)$ is a polynomial and $g(a) = f'(a)$). It follows that

$$(z^n + 1)\sum_{k=1}^{n} \frac{z - e^{2\pi ik/n}}{z + e^{2\pi ik/n}}$$

is polynomial of degree n, with leading coefficient n and it assumes value $-2n$ at $-e^{2\pi il/n}$ for all $1 \leq l \leq n$. The same property has the polynomial $n(z^n - 1)$ so

$$(z^n + 1)\sum_{k=1}^{n} \frac{z - e^{2\pi ik/n}}{z + e^{2\pi ik/n}} = n(z^n - 1).$$

We see that

$$a_1 + ... + a_n = in\frac{z^n - 1}{z^n + 1}$$

and

$$\frac{a_1 + a_2 + \cdots + a_n}{a_1 a_2 \cdots a_n} = \frac{in}{i^n} = (-1)^{(n-1)/2} n.$$

Remark: From the equality

$$a_k = \frac{e^{2i(\theta + k\pi/n)} - 1}{i(e^{2i(\theta + k\pi/n)} + 1)}$$

we get that

$$e^{2i(\theta + k\pi/n)} = \frac{i - a_k}{i + a_k}$$

so

$$\frac{(i - a_k)^n}{(i + a_k)^n} = e^{2in(\theta + k\pi/n)} = e^{2in\theta}$$

It follows that $a_1, ..., a_n$ are roots of the polynomial $e^{2in\theta}(i+x)^n - (i-x)^n$, i.e.

$$e^{2in\theta}(i+x)^n - (i-x)^n = (e^{2in\theta} + 1)(x - a_1)...(x - a_n).$$

Comparing the coefficient at x^{n-k} we see that

$$\sum_{i_1 < i_2 .. < i_k} a_{i_1} a_{i_2} ... a_{i_k} = (-1)^k \frac{\binom{n}{k} i^k (e^{2in\theta} - (-1)^{n-k})}{e^{2in\theta} + 1}.$$

Problem A6: Four points are chosen uniformly and independently at random in the interior of a given circle. Find the probability that they are the vertices of a convex quadrilateral.

Solution: Let A_1, A_2, A_3, A_4 be the four random points in in a unit circle (clearly we may assume that the circle has radius 1). The lines A_1A_2, A_2A_3 and A_3A_1 divide the circle into 7 parts. Note that $A_1A_2A_3A_4$ is convex iff A_4 is in one of the three parts which contain exactly one of the sides of the triangle $A_1A_2A_3$. So the probability that $A_1A_2A_3A_4$ is convex is equal to the expected area of these three parts divided by π (the area of the circle). Note that the expected area E of the triangle $A_1A_2A_3$ divided by π is the probability that A_r belongs to the triangle $A_1A_2A_3$. The expected area E_i of the part which intersects $A_1A_2A_3$ only at the vertex A_i divided by π is the probability that A_i is in the triangle $A_jA_kA_4$ (here $\{1,2,3\} = \{i,j,k\}$). Thus $E_1 = E_2 = E_3 = E$ and therefore the probability that $A_1A_2A_3A_4$ is convex is $(\pi - 4E)/\pi = 1 - 4E/\pi$. So it suffices to compute E.

Consider the set

$$\{S = (\alpha, r, a, b, x, y) : \alpha \in [0, 2\pi),\ 0 \le r \le 1,\ -\sqrt{1-r^2} \le a, b \le \sqrt{1-r^2},\ x^2+y^2 \le 1\}.$$

The map

$$F : (\alpha, r, a, b, x, y) \mapsto (a\cos\alpha + r\sin\alpha, -a\sin\alpha + r\cos\alpha, b\cos\alpha + r\sin\alpha, -b\sin\alpha + r\cos\alpha,$$

$$x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha)$$

is a parametrization of the product T of three unit circles (the inverse map corresponds to rotating the triangle with vertices A_1, A_2, A_3 inside the unit circle to a triangle $A_1'A_2'A_3'$ such that the line $A_1'A_2'$ is parallel to the x-axis and above it; here $A_1' = (a, r)$, $A_2' = (b, r)$, $A_3' = (x, y)$; α is the angle of rotation). For a point $t = (A_1, A_2, A_3)$ of T, let $A(t)$ be the area of $A_1A_2A_3$. Thus the expected area of a random triangle in the unit circle is

$$E = \pi^{-3}\int_T A = \pi^{-3}\int_S A \circ F|JF|$$

where JF is the Jacobian of F. Note that $A(F(\alpha, r, a, b, x, y)) = |a-b||y-r|/2$. Also it is not hard to see that $|JF| = |a-b|$ (this requires computation of a 6×6 determinant, but it is not hard). Thus

$$E = \pi^{-3}\frac{1}{2}\int_0^{2\pi}\int_0^1\int_D\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}}\int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}}(a-b)^2|r-y|\,da\,db\,dx\,dy\,dr\,d\alpha$$

where $D = \{x^2 + y^2 \le 1\}$ is the unit circle. Since the angle is not involved in the function under integral, after evaluation of the first two most-right integrals we get

$$E = \frac{8}{3}\pi^{-2}\int_0^1\int_D |r-y|(1-r^2)^2\,dx\,dy\,dr.$$

Now

$$\int_D |r-y|dx\,dy = \int_{-1}^{1} |r-y| \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = 2\int_{-1}^{1} |r-y|\sqrt{1-y^2}\,dy =$$

$$2[\int_{-1}^{r} (r-y)\sqrt{1-y^2}\,dy + \int_{r}^{1} (y-r)\sqrt{1-y^2}\,dy] = 4r\int_0^r \sqrt{1-y^2}\,dy + \frac{4}{3}(1-r^2)^{3/2}.$$

Thus

$$E = \frac{8}{3}\pi^{-2}\int_0^1 (1-r^2)^2[4r\int_0^r \sqrt{1-y^2}\,dy + \frac{4}{3}(1-r^2)^{3/2}]\,dr.$$

Since the derivative of $\int_0^r \sqrt{1-y^2}dy$ is $\sqrt{1-r^2}$ and $\frac{d}{dr}(1-r^2)^3 = -6r(1-r^2)^2$, integration by parts yields

$$\int_0^1 (1-r^2)^2 4r \int_0^r \sqrt{1-y^2}\,dy\,dr = \frac{2}{3}\int_0^1 (1-r^2)^{7/2}\,dr.$$

Thus

$$E = \frac{16}{3}\pi^{-2}\int_0^1 (1-r^2)^{7/2}dr = \frac{35}{48\pi}.$$

The probability that four random points in a circle form a convex quadrilateral equals then $1 - 35/12\pi^2$.

Problem B1: Show that the curve $x^3 + 3xy + y^3 = 1$ contains only one set of three distinct points, A, B, and C, which are vertices of an equilateral triangle, and find its area.

Solution: The key here is to realize that the curve is just a union of a line and a point. Indeed, note that

$$x^3+3xy+y^3-1 = (x+y-1)(x^2+y^2-xy+x+y+1) = (x+y-1)((x+1)^2+(y+1)^2+(x-y)^2)/2.$$

Thus our curve consists of the point $(-1,-1)$ and the line $x+y=1$. It is now clear that any equilateral triangle with vertices on the curve must have vertex $(-1,-1)$ and the side opposite this vertex has midpoint $(1/2,1/2)$. Thus, there is unique such triangle, its height is $h = 3\sqrt{2}/2$, its side is $a = 2h/\sqrt{3} = \sqrt{6}$ and its area is $ah/2 = 3\sqrt{3}/2$.

Problem B2: Prove that, for every set $X = \{x_1, x_2, \ldots, x_n\}$ of n real numbers, there exists a non-empty subset S of X and an integer m such that

$$\left| m + \sum_{s\in S} s \right| \le \frac{1}{n+1}.$$

Solution: Consider the intervals $[i/(n+1), (i+1)/(n+1)]$, $i = 0, 1, ..., n$. The problem asks as to show that the fractional part of $\sum_{s\in S} s$ belongs to either $[0, 1/(n+1)]$ or $[n/(n+1), 1]$ for some non-empty S. If the fractional part of any of the numbers x_1,

x_1+x_2, ..., $x_1+x_2+...+x_n$ is in one of the intervals $[0, 1/(n+1)]$ or $[n/(n+1), 1]$ we are done. Otherwise, each of these n fractional parts belongs to one of the remaining $n-1$ intervals. By pigeon-hole principle, there are $1 \leq k < m \leq n$ such that the fractional parts of both $x_1 + ... + x_k$ and $x_1 + ... + x_m$ belong to the same interval. It follows then that the fractional part of the difference $(x_1+...+x_m)-(x_1+...+x_k) = x_{k+1}+...+x_m$ is in one of the intervals $[0, 1/(n+1)]$ or $[n/(n+1), 1]$.

Problem B3: Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.

Solution: For a point x let L_S^x be the number of linear partitions of S obtained by lines through x. Consider a set S' of $n+1$ points and let $S' = S \cup \{x\}$ where S consists of n points. Each liner partition of S' restricts to a linear partition of S. This is surjective map from linear partitions of S' to linear partitions of S. If a linear partition of S can be obtained from 2 different linear partitions of S' then it can be realized by line through x and there are exactly 2 partitions of S' restricting to it. It follows that $L_{S'} = L_S + L_S^x$. Now for L_S^x rotate a line through x and note that you get at most n partitions of S this way (you change a partition when you pass through a point of S), and the equality holds if no three points of S' are collinear. Thus $L_{S'} \leq L_S + n$ and the equality holds if no three points of S' are collinear. A straightforward induction shows now that $L_S \leq 1 + \binom{n}{2}$ and equality holds if on three points of S are collinear.

Problem B4: Let Z_n denote the set of points in $\mathbb{R}^n$ whose coordinates are 0 or 1. (Thus Z_n has 2^n elements, which are the vertices of a unit hypercube in $\mathbb{R}^n$.) Given a vector subspace V of $\mathbb{R}^n$, let $Z_n(V)$ denote the number of members of Z_n that lie in V. Let k be given, $0 \leq k \leq n$. Find the maximum, over all vector subspaces $V \subseteq \mathbb{R}^n$ of dimension k, of the number of points in $V \cap Z_n$.)

Solution: Clearly the 2^k points whose first $n-k$ coordinates are 0 are in the k dimensional subspace of all points whose first $n-k$ coordinates are 0. Suppose we have 2^k+1 of our points in a k dimensional subspace V. Consider the projection $\mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ given by $(x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1})$. Clearly it maps Z_n onto Z_{n-1} and for each point of Z_{n-1} there are exactly 2 points of Z_n mapped to it. If this projection is injective on V then the image of V is a k-dimensional subspace of $\mathbb{R}^{n-1}$ which contains 2^k+1 points of Z_{n-1}. If the projection is not injective then the image of V has dimension $k-1$ and the image of $Z_n \cap V$ contains at least $2^{k-1}+1$ different points of Z_{n-1}. Continuing this process we get an l dimensional subspace of $\mathbb{R}$ which contains at lest 2^l+1 elements of Z_1. This is clearly false, so a k dimensional subspace of $\mathbb{R}^n$ contains at most 2^k elements of Z_n.

Problem B5: For each continuous function $f : [0, 1] \to \mathbb{R}$, let $I(f) = \int_0^1 x^2 f(x)\,dx$ and $J(x) = \int_0^1 x\,(f(x))^2\,dx$. Find the maximum value of $I(f) - J(f)$ over all such functions f.

Solution: Recall that $ab \le (a+b)^2/4$ for any real numbers a, b. In particular, $f(x)(x-f(x)) \le x^2/4$ for all x. Thus

$$I(f)-J(f) = \int_0^1 (x^2 f(x) - x\,(f(x))^2)\,dx = \int_0^1 x f(x)(x-f(x))\,dx \le \int_0^1 x^3/4\,dx = 1/16.$$

and the equality holds for $f(x) = x/2$.

Problem B6: Let k be an integer greater than 1. Suppose that $a_0 > 0$ and define

$$a_{n+1} = a_n + \frac{1}{\sqrt[k]{a_n}}$$

for $n \ge 0$. Evaluate

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k}.$$

Solution: Let $f(x) = x + x^{-1/k}$. Clearly $f(x) > 1$, f is increasing on $(1, \infty)$ and $f(x) > 2$ for $x > 1$. Let $c = (k + 1/k)^{k/k+1}$. We claim that $f(cx^{k/k+1}) \ge c(x+1)^{k/k+1}$. In fact, this is equivalent to

$$c^{\frac{k+1}{k}} x + 1 = \frac{k+1}{k}(x + \frac{k}{k+1}) \ge \frac{k+1}{k} x^{\frac{1}{k+1}} (x+1)^{\frac{k}{k+1}},$$

i.e. to

$$x + \frac{k}{k+1} \ge x^{\frac{1}{k+1}} (x+1)^{\frac{k}{k+1}}.$$

This follows from the generalized AMGM inequality:

$$x^{\frac{1}{k+1}} (x+1)^{\frac{k}{k+1}} \le \frac{1}{k+1} x + \frac{k}{k+1}(x+1) = x + \frac{k}{k+1}.$$

It is clear that a_n increases and hence tends to infinity (pass to the limit in the recursion formula for a_n). There is an m such that $a_m > c$ and then easy induction and the inequality above yield that

$$a_{m+n} \ge c(n+1)^{k/k+1} \quad (*).$$

On the other hand, using $(*)$ we get

$$a_{m+n} = a_{m+1} + \sum_{i=1}^{n-1} \frac{1}{a_{m+i}^{1/k}} \le a_{m+1} + \frac{1}{c^{1/k}} \sum_{i=2}^{n} \frac{1}{i^{1/(k+1)}} \le a_{m+1} + \frac{1}{c^{1/k}} \int_1^n \frac{dx}{x^{1/(k+1)}} \le$$

$$\le a_{m+1} + \frac{1}{c^{1/k}} \frac{k+1}{k} (n^{k/k+1} - 1) \le a_{m+1} + cn^{k/k+1}$$

(since $\frac{1}{c^{1/k}} \frac{k+1}{k} = c$). Thus we proved that

$$a_{m+1} + cn^{k/k+1} \ge a_{m+n} \ge c(n+1)^{k/k+1}.$$

Pinching theorem yields now that

$$\lim_{n\to\infty} \frac{a_n}{n^{k/k+1}} = c$$

and therefore

$$\lim_{n\to\infty} \frac{a_n^{k+1}}{n^k} = c^{k+1} = (\frac{k+1}{k})^k.$$

Remark 1 Set $b_n = a_{m+n}/n^{k/k+1}$. We claim that b_n is decreasing. In fact,

$$b_{n+1}/b_n = (1 + a_{m+n}^{-(k+1)/k})(\frac{n}{n+1})^{k/(k+1)} \leq 1$$

is equivalent to

$$a_{m+n}^{(k+1)/k} \geq \frac{n^{k/k+1}}{(n+1)^{k/k+1} - n^{k/k+1}}.$$

Since the mean value theorem yields $(n+1)^{k/k+1} - n^{k/k+1} = (k/(k+1))u^{-1/(k+1)}$ for some $n \leq u \leq n+1$ and $n^{k/k+1}u^{1/(k+1)} \leq n+1$, it suffices to show that

$$a_{m+n}^{(k+1)/k} \geq \frac{k+1}{k}(n+1),$$

which is equivalent to $(*)$.

Remark 2 The limit can be computed easily using the following Stolz Theorem (an analog of L'Hospitals rule for sequences)

Stolz Theorem *If a_n, b_n are sequences such that b_n is increasing and unbounded and*

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = g$$

exists, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = g.$$

Let us apply it to our sequence $a_n^{k+1/k}$ and $b_n = n$ so the problem reduces to finding the limit of $a_{n+1}^{k+1/k} - a_n^{k+1/k}$. The Mean Value theorem for the function $x^{k+1/k}$ implies that

$$a_{n+1}^{k+1/k} - a_n^{k+1/k} = \frac{k+1}{k} u_n^{1/k}(a_{n+1} - a_n) = \frac{k+1}{k} u_n^{1/k} a_n^{-1/k}$$

for some $a_n \leq u_n \leq a_{n+1}$. Since a_{n+1}/a_n converges to 1, we see that u_n/a_n converges to 1 and therefore

$$\lim_{n\to\infty} a_n^{k+1/k}/n = \lim_{n\to\infty} (a_{n+1}^{k+1/k} - a_n^{k+1/k}) = \frac{k+1}{k}.$$