Solutions to Exam 1

Problem 1. Compute the integerals

$$\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx.$$

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx.$$

Solution: Do the substitution $x = \pi - y$ and use the fact that $\sin(\pi - a) = \sin a$ to get

$$\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx = \int_0^\pi \frac{(\pi - x)^2 \sin x}{x^2 + (\pi - x)^2} dx$$

and

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \int_0^\pi \frac{(\pi - x)^3 \sin x}{3(\pi - x)^2 - 3\pi(\pi - x) + \pi^2} dx.$$

It follows that

$$2\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx = \int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx + \int_0^\pi \frac{(\pi - x)^2 \sin x}{x^2 + (\pi - x)^2} dx =$$

$$\int_0^\pi \frac{(x^2 + (\pi - x)^2) \sin x}{x^2 + (\pi - x)^2} dx = \int_0^\pi \sin x dx = (-\cos \pi) - (-\cos 0) = 2.$$

Thus

$$\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx = 1.$$

For the second integral note that

$$3(\pi - x)^2 - 3\pi(\pi - x) + \pi^2 = 3x^2 - 3\pi x + \pi^2.$$

It follows that

$$2\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx + \int_0^\pi \frac{(\pi - x)^3 \sin x}{3(\pi - x)^2 - 3\pi(\pi - x) + \pi^2} dx =$$

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx + \int_0^\pi \frac{(\pi - x)^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx + = \int_0^\pi \frac{[x^3 + (\pi - x)^3] \sin x}{3x^2 - 3\pi x + \pi^2} dx.$$

Note now that $x^3 + (\pi - x)^3 = \pi(3x^2 - 3\pi x + \pi^2)$, so

$$2\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \int_0^\pi \pi \sin x dx = \pi[(-\cos \pi) - (-\cos 0)] = 2\pi.$$

Thus

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \pi.$$

Problem 2. A set of 10 different integers is selected from $\{1, 2, 3, ..., 18\}$. Prove that among the selected integers there are two numbers which differ by 3.

Solution: Consider the following nine sets:

$$\{1, 4\}, \{2, 5\}, \{3, 6\}, \{7, 10\}, \{8, 11\}, \{9, 12\}, \{13, 16\}, \{14, 17\}, \{15, 18\}.$$

Each of the 10 chosen numbers belongs to exactly one of the nine sets. By the pigeon-hole principle, some two of the chosen numbers are in the same set, so they differ by 3.

Problem 3. Positive numbers a, b, c satisfy $a^{-1} + b^{-1} + c^{-1} = 3$.

a) Prove that $abc \geq 1$;

b) Prove that $(a + b)(a + c)(b + c) \geq 8$. When does the equality hold?

Solution: a) Note that the arithemtic mean of a^{-1}, b^{-1}, c^{-1} is 1, so the geometric mean $\sqrt[3]{a^{-1}b^{-1}c^{-1}} \leq 1$, which is equivalent to $abc \geq 1$.

Alternatively, note that the harmonic mean of a, b, c is

$$M_{-1}(a, b, c) = \frac{3}{a^{-1} + b^{-1} + c^{-1}} = 1.$$

Thus $M_0(a, b, c) = \sqrt[3]{abc} \geq M_{-1}(a, b, c) = 1$, i.e. $abc \geq 1$.

b) We have $a + b \geq 2\sqrt{ab}$, $b + c \geq 2\sqrt{bc}$, $a + c \geq 2\sqrt{ac}$. It follows that

$$(a + b)(a + c)(b + c) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ac}) = 8abc.$$

By a) we see now that $(a + b)(a + c)(b + c) \geq 8$. The equality holds iff all three inequalities above are equalities and $abc = 1$, i.e. $a = b$, $b = c$, $c = a$, $abc = 1$. Thus the equality holds iff $a = b = c = 1$.

Problem 4. Numbers $1, 2, ..., 2007$ are written on a blackboard. Every now and then somebody picks two numbers a and b and replaces them by $a - 3$, $b + 1$. Is it possible that at some point all the numbers on the blackboard are odd? Justify your answer.

Solution: Note that among the numbers $1, 2, ..., 2007$ we have 1003 even numbers and 1004 odd numbers. Since both substitutions $a \mapsto a-3$ and $b \mapsto b+1$ change parity, the number of odd numbers on the balckboard before and after substitution is either the same (when a and b have different parity), or increases by 2 (if both a and b are even), or decreases by 2 (if both a and b are odd). Since we start with an even number of odd numbers, the number of odd numbers will remain even all the time. Thus we can not have all 2007 numbers odd at any time.

Alternatively, look at sum S of the numbers on the blackboard. Each time we make a change, the sum goes down by 2. So the parity of the sum is always the same. Since $1+2+...+2007 = 2007 \cdot 1004$ is even, the sum of the numbers on the blackboard is always even. But the sum of 2007 odd numbers is odd, so we can not have all 2007 numbers odd at any time.