Homework

due on Wednesday, November 17

Problem 1. Initially, we are given the sequence $1, 2, ...100$. Every minute, we erase two numbers u and v and replace them with the value $uv + u + v$. Clearly, we will be left with just one number after 99 minutes. What is this number? Justify your answer.

Problem 2. Start with the set $\{3, 4, 12\}$. You are then allowed to replace any two numbers a and b with the new pair $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you transform the set into $\{4, 6, 12\}$? Look for an invariant.

Problem 3. Consider an $m \times n$ table of integers. If the sum of all elements in a row or column is negative you may change the sign of all numbers in that row (or column). Prove that at some point the sum of elements in every row and column will be nonnegative. Look for a monovariant (something which increases in each step and is bounded). Prove that the problem remains true if the numbers in the table are any real numbers.

Problem 4. A 23×23 square is tiled by 1×1, 2×2, and 3×3 tiles. Prove that at least one 1×1 tile must be used . Find such a tiling with exactly one 1×1 tile. Hint: put a number in each 1×1 square of the big sqaure so that 2×2 and 3×3 tiles cover a total divisible by 3.

Problem 5. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well. Hint. Walk through the house locking doors.

Problem 6. Consider an $n \times n$ chessboard with all 4 corner squares removed. Prove that if the board can be covered with $L-$tetrominoes then $n - 2$ is a multiple of 4. Is the converse true? (an $L-$tetromino is a plane figure shown below, constructed from four unit squares arranged in the form of L)