Solutions to Exam 1

Problem 1. Prove that there is no integer $n > 2$ such that $n(n+6)$ is a square of an integer.

Solution. Note that for any n we have

$$(n+3)^2 = n^2 + 6n + 9 > n^2 + 6n = n(n+6).$$

On the other hand,

$$(n+2)^2 = n^2 + 4n + 4 = n^2 + 6n + 4 - 2n = n(n+6) - 2(n-2) < n(n+6)$$

provided $n - 2 > 0$, i.e. $n > 2$. Thus, for $n > 2$, the number $n(n+6)$ is contained between two consecutive squares $(n+2)^2$ and $(n+3)^2$, hence it is not a square.

Problem 2. A set of 10 different integers is selected from $\{1, 2, 3, ..., 18\}$. Prove that among the selected integers there are two numbers which differ by 3.

Solution: Consider the following nine sets:

$$\{1,4\}, \{2,5\}, \{3,6\}, \{7,10\}, \{8,11\}, \{9,12\}, \{13,16\}, \{14,17\}, \{15,18\}.$$

Each of the 10 chosen numbers belongs to exactly one of the nine sets. By the pigeon-hole principle, some two of the chosen numbers are in the same set, so they differ by 3.

Problem 3. Let n be a positive integer. Let $d_1, \ldots, d_k$ be all divisors of n. Prove that the number

$$\frac{2}{\ln n} \sum_{i=1}^{k} \ln d_i = \frac{2}{\ln n}(\ln d_1 + \ln d_2 + \ldots + \ln d_k)$$

is an integer.

Solution. The main observation is that the divisors of n come in pairs: if d is a divisor then so is n/d. Thus the sequences $n/d_1, n/d_2, \ldots, n/d_k$ and $d_1, \ldots, d_k$ both list all divisors of n (just in different order). We may then write

$$\ln d_1 + \ln d_2 + \ldots + \ln d_k = \ln \frac{n}{d_1} + \ln \frac{n}{d_2} + \ldots + \ln \frac{n}{d_k}$$

and consequently

$$2(\ln d_1 + \ln d_2 + \ldots + \ln d_k) = (\ln d_1 + \ln \frac{n}{d_1}) + (\ln d_2 + \ln \frac{n}{d_2}) + \ldots + (\ln d_k + \ln \frac{n}{d_k}).$$

Observe now that $\ln d + \ln \frac{n}{d} = \ln n$ for any d. It follows that $2(\ln d_1 + \ln d_2 + \ldots + \ln d_k) = k \ln n$ and therefore

$$\frac{2}{\ln n}(\ln d_1 + \ln d_2 + \ldots + \ln d_k) = k$$

is an integer.

Problem 4. Compute the integerals

$$\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx.$$

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx.$$

Solution: Do the substitution $x = \pi - y$ and use the fact that $\sin(\pi - a) = \sin a$ to get

$$\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx = \int_0^\pi \frac{(\pi - x)^2 \sin x}{x^2 + (\pi - x)^2} dx$$

and

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \int_0^\pi \frac{(\pi - x)^3 \sin x}{3(\pi - x)^2 - 3\pi(\pi - x) + \pi^2} dx.$$

It follows that

$$2\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx = \int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx + \int_0^\pi \frac{(\pi - x)^2 \sin x}{x^2 + (\pi - x)^2} dx =$$

$$\int_0^\pi \frac{(x^2 + (\pi - x)^2) \sin x}{x^2 + (\pi - x)^2} dx = \int_0^\pi \sin x dx = (-\cos \pi) - (-\cos 0) = 2.$$

Thus

$$\int_0^\pi \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx = 1.$$

For the second integral note that

$$3(\pi - x)^2 - 3\pi(\pi - x) + \pi^2 = 3x^2 - 3\pi x + \pi^2.$$

It follows that

$$2\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx + \int_0^\pi \frac{(\pi - x)^3 \sin x}{3(\pi - x)^2 - 3\pi(\pi - x) + \pi^2} dx =$$

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx + \int_0^\pi \frac{(\pi - x)^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx + = \int_0^\pi \frac{[x^3 + (\pi - x)^3] \sin x}{3x^2 - 3\pi x + \pi^2} dx.$$

Note now that $x^3 + (\pi - x)^3 = \pi(3x^2 - 3\pi x + \pi^2)$, so

$$2 \int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \int_0^\pi \pi \sin x dx = \pi[(-\cos \pi) - (-\cos 0)] = 2\pi.$$

Thus

$$\int_0^\pi \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx = \pi.$$