Solutions to Exam 1

Problem 1. Prove that there is no integer $n > 2$ such that $n(n+6)$ is a square of an integer.

Solution. Note that for any n we have

$$(n+3)^2 = n^2 + 6n + 9 > n^2 + 6n = n(n+6).$$

On the other hand,

$$(n+2)^2 = n^2 + 4n + 4 = n^2 + 6n + 4 - 2n = n(n+6) - 2(n-2) < n(n+6)$$

provided $n - 2 > 0$, i.e. $n > 2$. Thus, for $n > 2$, the number $n(n+6)$ is contained between two consecutive squares $(n+2)^2$ and $(n+3)^2$, hence it is not a square.

Problem 2. Compute the integral

$$\int_0^1 \frac{(2x^2+1)\sin \pi x}{x^2 - x + 1} dx.$$

Be very careful with your algebra. Note: $\sin(\pi - t) = \sin t$.

Solution. Do the substitution $x = 1 - y$ and use the fact that $\sin \pi(1-y) = \sin \pi y$ and $(1-y)^2 - (1-y) + 1 = y^2 - y + 1$ to get

$$\int_0^1 \frac{(2x^2+1)\sin \pi x}{x^2 - x + 1} dx = \int_0^1 \frac{(2(1-y)^2+1)\sin \pi y}{y^2 - y + 1} dy = \int_0^1 \frac{(2x^2 - 4x + 3)\sin \pi x}{x^2 - x + 1} dx.$$

(we replaced y by x in the last integral). It follows that

$$2\int_0^1 \frac{(2x^2+1)\sin \pi x}{x^2 - x + 1} dx = \int_0^1 \frac{(2x^2+1)\sin \pi x}{x^2 - x + 1} dx + \int_0^1 \frac{(2x^2 - 4x + 3)\sin \pi x}{x^2 - x + 1} dx =$$

$$= \int_0^1 \frac{(4x^2 - 4x + 4)\sin \pi x}{x^2 - x + 1} dx = 4\int_0^1 \sin \pi x dx = \frac{-4}{\pi}(\cos \pi - \cos 0) = \frac{8}{\pi}$$

Problem 3. Let n be a positive integer. Let $d_1, \ldots, d_k$ be all divisors of n. Prove that the number

$$\frac{2}{\ln n} \sum_{i=1}^{k} \ln d_i = \frac{2}{\ln n}(\ln d_1 + \ln d_2 + \ldots + \ln d_k)$$

is an integer.

Solution. The main observation is that the divisors of n come in pairs: if d is a divisor then so is n/d. Thus the sequences $n/d_1, n/d_2, \ldots, n/d_k$ and $d_1, \ldots, d_k$ both list all divisors of n (just in different order). We may then write

$$\ln d_1 + \ln d_2 + \ldots + \ln d_k = \ln \frac{n}{d_1} + \ln \frac{n}{d_2} + \ldots + \ln \frac{n}{d_k}$$

and consequently

$$2(\ln d_1 + \ln d_2 + \ldots + \ln d_k) = (\ln d_1 + \ln \frac{n}{d_1}) + (\ln d_2 + \ln \frac{n}{d_2}) + \ldots + (\ln d_k + \ln \frac{n}{d_k}).$$

Observe now that $\ln d + \ln \frac{n}{d} = \ln n$ for any d. It follows that $2(\ln d_1 + \ln d_2 + \ldots + \ln d_k) = k \ln n$ and therefore

$$\frac{2}{\ln n}(\ln d_1 + \ln d_2 + \ldots + \ln d_k) = k$$

is an integer.

Problem 4. Compute the sum

$$\sum_{k=0}^{2010} \frac{\sin^{2011} \frac{k\pi}{4020}}{\sin^{2011} \frac{k\pi}{4020} + \cos^{2011} \frac{k\pi}{4020}}.$$

Solution: Note that if k changes from 0 to 2010 then $2010 - k$ changes from 2010 to 0. Thus we have

$$\sum_{k=0}^{2010} \frac{\sin^{2011} \frac{k\pi}{4020}}{\sin^{2011} \frac{k\pi}{4020} + \cos^{2011} \frac{k\pi}{4020}} = \sum_{k=0}^{2010} \frac{\sin^{2011} \frac{(2010-k)\pi}{4020}}{\sin^{2011} \frac{(2010-k)\pi}{4020} + \cos^{2011} \frac{(2010-k)\pi}{4020}}.$$

(this is analogous to change of variables in integrals; we simply are summing the terms from the end). Note that

$$\sin \frac{(2010-k)\pi}{4020} = \sin\left(\frac{\pi}{2} - \frac{k\pi}{4020}\right) = \cos \frac{k\pi}{4020}$$

and similarly

$$\cos \frac{(2010-k)\pi}{4020} = \cos\left(\frac{\pi}{2} - \frac{k\pi}{4020}\right) = \sin \frac{k\pi}{4020}.$$

It follows that

$$\sum_{k=0}^{2010} \frac{\sin^{2011} \frac{(2010-k)\pi}{4020}}{\sin^{2011} \frac{(2010-k)\pi}{4020} + \cos^{2011} \frac{(2010-k)\pi}{4020}} = \sum_{k=0}^{2010} \frac{\cos^{2011} \frac{k\pi}{4020}}{\sin^{2011} \frac{k\pi}{4020} + \cos^{2011} \frac{k\pi}{4020}}.$$

Thus

$$2\sum_{k=0}^{2010}\frac{\sin^{2011}\frac{k\pi}{4020}}{\sin^{2011}\frac{k\pi}{4020}+\cos^{2011}\frac{k\pi}{4020}}=$$

$$=\sum_{k=0}^{2010}\frac{\sin^{2011}\frac{k\pi}{4020}}{\sin^{2011}\frac{k\pi}{4020}+\cos^{2011}\frac{k\pi}{4020}}+\sum_{k=0}^{2010}\frac{\cos^{2011}\frac{k\pi}{4020}}{\sin^{2011}\frac{k\pi}{4020}+\cos^{2011}\frac{k\pi}{4020}}=$$

$$\sum_{k=0}^{2010}\frac{\sin^{2011}\frac{k\pi}{4020}+\cos^{2011}\frac{k\pi}{4020}}{\sin^{2011}\frac{k\pi}{4020}+\cos^{2011}\frac{k\pi}{4020}}=2011$$

and therefore

$$\sum_{k=0}^{2010}\frac{\sin^{2011}\frac{k\pi}{4020}}{\sin^{2011}\frac{k\pi}{4020}+\cos^{2011}\frac{k\pi}{4020}}=\frac{2011}{2}$$

Problem 5. There are 28 points selected in an equilateral triangle with side of length 3. Prove that there are 4 among these points which are within distance 1 to each other.

Solution: Divide the triangle into 9 equilateral triangles with side of length 1 (by dividing each side of the triangle into 3 equal pieces and drawing through the points of division lines parallel to the sides of the triangle). By the pigeon-hole principle, one of the small triangles must contain at least $\lceil\frac{28}{9}\rceil = 4$ of the selected points and then any two of these 4 points are no further than 1 apart.

Problem 6. A set of 10 different numbers is selected from $\{1, 2, \dots, 18\}$. Prove that among the selected integers there are two numbers which differ by 3.

Solution: Consider the following nine sets:

$$\{1,4\},\{2,5\},\{3,6\},\{7,10\},\{8,11\},\{9,12\},\{13,16\},\{14,17\},\{15,18\}.$$

Each of the 10 chosen numbers belongs to exactly one of the nine sets. By the pigeon-hole principle, some two of the chosen numbers are in the same set, so they differ by 3.