

Homework
due on Monday, May 3

Read carefully chapters 6 and 7 of Dunham's book and Chapters 8,10,11 and 17 in the book by Berlinghoff and Gouvea. Do question 2 of Chapter 8, question 4 of Chapter 10, question 3 of Chapter 11. Also solve the following problems.

Problem 1. Consider a cubic equation $x^3 + px + q = 0$.

- a) Show that if $p \geq 0$ then this equation has unique real solution (use calculus).
- b) Suppose now that $p < 0$. Prove that the function $f(x) = x^3 + px + q$ has a local maximum at $x_1 = -\sqrt{-p/3}$ and a local minimum at $x_2 = \sqrt{-p/3}$. Conclude that $f(x) = 0$ has more than one real solution iff $f(x_1) \geq 0$ and $f(x_2) \leq 0$.
- c) Use b) to show that if $p < 0$ then $x^3 + px + q = 0$ has more than one real solution iff

$$\Delta = \frac{p^3}{27} + \frac{q^2}{4} \leq 0$$

and has three distinct solutions iff $\Delta < 0$.

Problem 2. This problem describes Viète's approach to cubic equations which have three real roots. Let $x^3 + px + q = 0$ be such an equation. We know from the previous problem that $p < 0$. Let $R = \sqrt{-p/3}$.

- a) Use b) or c) of the previous problem to show that

$$-1 < \frac{q}{2R^3} < 1.$$

Conclude that

$$\frac{-q}{2R^3} = \cos \phi$$

for some $\phi \in (0, \pi)$.

- b) Prove that $\cos(3a) = 4\cos^3(a) - 3\cos(a)$ for every a (use the well known formulas for $\cos(a+b)$, $\cos(2a)$ and $\sin(2a)$).
- c) For $i = 0, 1, 2$ let

$$\phi_i = \frac{\phi}{3} + i\frac{2\pi}{3},$$

where ϕ is defined in part a). Prove that $x_i = 2R \cos \phi_i$ is a root of $x^3 + px + q = 0$ for $i = 0, 1, 2$ and that these roots are distinct.

d) Show that $x_1 = -R[\cos(\phi/3) + \sqrt{3} \sin(\phi/3)]$ and $x_2 = -R[\cos(\phi/3) - \sqrt{3} \sin(\phi/3)]$.

e) Use Viete's method to solve $x^3 - 9x + 3\sqrt{6} = 0$. Then use Cardano's formula to solve it. What can you say?