Homework

due on Thursday, May 8

Read carefully chapter 11 of Joseph's book and the remaining chapters of Dunham's book. Solve the following problems.

Problem 1. a) Explain how to use Ptolemy's theorem to prove that

$$\sin(x-y) = \sin x \cos y - \sin y \cos x.$$

b) What is the ratio of the diagonal in a regular pentagon to its side? (Euclid, Book XIII, Prop. 8). Explain how this implies that

$$\sin 72^\circ = \frac{\sqrt{5}+1}{2}\sin 36^\circ.$$

Use this equality and the formula for $\sin 2x$ to show that $\cos 36^\circ = (\sqrt{5}+1)/4$. Conclude that $\sin 36^\circ = \sqrt{10-2\sqrt{5}}/4$.

c) Use b) to prove that $\sin 18^\circ = (\sqrt{5}-1)/4$ and $\cos 18^\circ = \sqrt{10+2\sqrt{5}}/4$. Show that $\sin 15^\circ = (\sqrt{6}-\sqrt{2})/4$ and $\cos 15^\circ = (\sqrt{6}+\sqrt{2})/4$ (hint: $15 = 45 - 30$). Then show that

$$\sin 3^\circ = \frac{\sqrt{30}+\sqrt{10}-\sqrt{6}-\sqrt{2}+2\sqrt{5+\sqrt{5}}-2\sqrt{15+3\sqrt{5}}}{16}$$

and

$$\cos 3^\circ = \frac{\sqrt{30}-\sqrt{10}-\sqrt{6}+\sqrt{2}+2\sqrt{5+\sqrt{5}}+2\sqrt{15+3\sqrt{5}}}{16}.$$

Hint: $3 = 18 - 15$, and $\cos(x-y) = \cos x \cos y + \sin x \sin y$.

Problem 2. Learn a proof of the Theorem of Menalaus (for example, from link provided on the course web page). Use this theorem to prove the following results

a) The lines AA_1, BB_1 and CC_1 intersect in one point O. Let the lines AB and A_1B_1 intersect at C_2, the lines AC and A_1C_1 intersect at B_2, and the lines BC and B_1C_1 intersect at A_2. Prove that the points A_2, B_2, C_2 are collinear. This is often called Desarques Theorem. Hint. Apply Menelaus' Theorem to triangles OAB, OBC, OAC and appropriate lines. Then apply its converse to the triangle ABC.

b) Points A_1, B_1, C_1 are collinear and so are points A_2, B_2, C_2. The lines A_1B_2 and A_2B_1 intesect at a point C, the lines A_1C_2 and A_2C_1 intesect at a point B, and the lines B_1C_2 and B_2C_1 intesect at a point A. Prove that the points A, B, C are collinear. This is Pappus' Theorem. Hint. Let A_0, B_0, C_0 be the vertices of the triangle determined by the lines A_1B_2, B_1C_2, and C_1A_2 (where A_0 is the point of intersection of A_1B_2 and A_2C_1, etc.). Apply Menalaus' Theorem to the triangle $A_0B_0C_0$ and five appropriate lines.

Problem 3. Justify Newton's solution of the duplication of a cube, as described in the link on the course web page, by following the steps below (the notation is from the link). Alternatively, find your own justification.

a) Let $AG = x$, $CG = y$. Show that $\angle ACG = \pi/2$ and derive a relation between x and y.

b) Show that $DC = \sqrt{3}$.

c) Apply the Theorem of Menelaus to the triangle ADG. This should give you another relation between x and y.

d) Find x and y.

Problem 4. Let $ABCD$ be a convex quadrilateral. Let F be the point of intersection of its diagonals AC and BD. Set $a = AB$, $b = BC$, $c = CD$, $d = DA$, $p = (a+b+c+d)/2$, $e = AC$, $f = BD$, $\alpha = \angle AFB$.

a) Prove that $a^2 + c^2 - b^2 - d^2 = -2ef\cos\alpha$. Hint: Apply the law of cosines to triangles AFB, BFC, CFD, DFA.

b) Prove that the area S of the quadrilateral is given by $2S = ef\sin\alpha$. Hint: Add areas of the triangles AFB, BFC, CFD, DFA.

c) Show that $4(p-a)(p-c) = b^2 + d^2 - a^2 - c^2 + 2(ac+bd)$ and $4(p-b)(p-d) = a^2 + c^2 - b^2 - d^2 + 2(ac+bd)$. Conclude that $16(p-a)(p-b)(p-c)(p-d) = 4[(ac+bd)^2 - (ef)^2] + 16S^2$. Hint: Use a) and b).

d) Prove that if $ABCD$ is cyclic then $S = \sqrt{(p-a)(p-b)(p-c)(p-d)}$. Hint: Use Ptolemy's Theorem.

The formula in d) generalizes Heron's Theorem. It has been stated by an Indian mathematician and astronomer Brahmagupta (598-670 AD), but we do not know how he derived this formula.

Remark: One can prove that $(ac)^2 + (bd)^2 - (ef)^2 = 2abcd \cos 2\theta$, where $2\theta = \angle DAB + \angle BCD$. Using this one can show that $S^2 = (p-a)(p-b)(p-c)(p-d) - abcd \cos^2 \theta$.

Problem 5. Consider a cubic equation $x^3 + px + q = 0$.

a) Show that if $p \geq 0$ then this equation has unique real solution (use calculus).

b) Suppose now that $p < 0$. Prove that the function $f(x) = x^3 + px + q$ has a local maximum at $x_1 = -\sqrt{-p/3}$ and a local minimum at $x_2 = \sqrt{-p/3}$. Conclude that $f(x) = 0$ has more than one real solution iff $f(x_1) \geq 0$ and $f(x_2) \leq 0$.

c) Use b) to show that if $p < 0$ then $x^3 + px + q = 0$ has more than one real solution iff

$$\Delta = \frac{p^3}{27} + \frac{q^2}{4} \leq 0$$

and has three distinct real solutions iff $\Delta < 0$.

Problem 6. This problem describes Viete's approach to cubic equations which have three real roots. Let $x^3+px+q = 0$ be such an equation. We know from the previous problem that $p < 0$. Let $R = \sqrt{-p/3}$.

a) Use b) or c) of the previous problem to show that

$$-1 < \frac{q}{2R^3} < 1.$$

Conclude that

$$\frac{-q}{2R^3} = \cos \phi$$

for some $\phi \in (0, \pi)$.

b) Prove that $\cos(3a) = 4\cos^3(a) - 3\cos(a)$ for every a (use the well known formulas for $\cos(a+b)$, $\cos(2a)$ and $\sin(2a)$). c) For $i = 0, 1, 2$ let

$$\phi_i = \frac{\phi}{3} + i\frac{2\pi}{3},$$

where ϕ is defined in part a). Prove that $x_i = 2R\cos\phi_i$ is a root of $x^3 + px + q = 0$ for $i = 0, 1, 2$ and that these roots are disctinct.

d) Show that $x_1 = -R[\cos(\phi/3) + \sqrt{3}\sin(\phi/3)]$ and $x_2 = -R[\cos(\phi/3) - \sqrt{3}\sin(\phi/3)]$.

e) Use Viete's method to solve $x^3 - 9x + 3\sqrt{6} = 0$. Then use Cardano's formula to solve it. What can you say?