Homework

due on Thursday, February 18

Read carefully Chapters 3 and 4 of the book by Joseph. Solve the following problems.

Problem 1. a) Write $7217\frac{2}{15}$ using Babylonian sexagesimal notation and Egyptian hieroglyphic notation.

b) In class we found that in sexagesimal number system we have

$$\frac{1}{7} = 0; 08, 34, 17, 08, 34, 17, 08, \ldots.$$

Express $\dfrac{1}{11}$ in sexagesimal system (show your work).

Problem 2. One possible method the Babylonians might have used to approximate the square root of b is the following: start with an approximation a_1 to $\sqrt{b}$. Then take for a new approximation $a_2 = (a_1 + b/a_1)/2$. And keep doing it (the Babylonians would do one, perhaps two steps), i.e you define recursively a sequence of approximations by the formula

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{b}{a_n}\right).$$

a) Starting with $b = 5$ and $a_1 = 2$ compute a_3 in sexagesimal system and in decimal system. How many decimal digits agree with the actual value of $\sqrt{5}$?

b) In general, assuming that the sequence a_n converges, prove that its limit is $\sqrt{b}$.

c) (Extra credit). Prove that $a_n \geq \sqrt{b}$ for all $n \geq 2$ (hint: prove first that $x + y \geq 2\sqrt{xy}$ for any positive x, y). Then prove by induction that $a_{n+1} \leq a_n$ for all $n \geq 2$ and conclude that a_n converges. This proves that the sequence a_n consists indeed of better and better aproximations to $\sqrt{b}$.

Remark. Suppose for simplicity that $b > 1$ and that $a_2 - \sqrt{b} < 1$. It is easy to see that

$$a_{n+1} - \sqrt{b} = \frac{1}{2a_n}(a_n - \sqrt{b})^2.$$

Since $a_n \geq \sqrt{b}$ for all $n \geq 2$, it is not hard to prove by induction that

$$a_n - \sqrt{b} \leq \frac{1}{(2\sqrt{b})^{2^{n-2}-1}}.$$

It follows that a_n converges to $\sqrt{b}$ very fast.

Problem 3. a) Use the greedy algorithm to find Egyptian fraction for 24/29.

b) (Extra credit) Find an Egyptian fraction for 24/29 with all the denominators even and less than 250. Explain how you did it.

Problem 4. Assume the following result: the volume of a pyramid (or cone) with base of area A (any shape) and height H is equal to $AH/3$. Suppose now that such a pyramid is truncated by a plane parallel to the base whose distance to the base is h and let B be the area of the top of the truncated pyramid. Prove that the volume V of the truncated pyramid is given by the following formula:

$$V = \frac{1}{3}h(A + \sqrt{AB} + B).$$

Explain carefully your argument using full sentences. Hint. Use the facts that the two pyramids determined by the top and bottom planes are similar.

Problem 5. Consider a trapezoid (also called trapezium outside US) with parallel sides of lengths a, b and height h (assume $a < b$). A line parallel to the sides divides the trapezoid into two parts of equal area. Find the distance d between the longer (parallel) side and the line. Hint: Let $f = (b-a)/h$. Consider any line parallel to the sides which divides the trapezoid into 2 smaller trapezoids. Observe that each of the smaller trapezoids and the original trapezoid all have the same value of f. Use this to show first that the length c of the segment which bisects the trapezoid satisfies $c^2 = (a^2 + b^2)/2$ (this problem and a solution can be found on some Babylonian clay tablets).