Homework

due on Thursday, March 17

Read carefully chapters 3,4 of Dunham's book. Solve the following problems.

Problem 1. a) Let n be a positive integer of the form $4k-1$. Prove that n has a prime factor of the same form (you may assume that every natural number > 1 is a product of prime numbers). Hint: What can you say about product of numbers of the form $4k+1$? Note that every odd number is either of the form $4k+1$ or $4k-1$.

b) Prove that there are infinitely many prime numbers of the form $4k-1$. Hint: Modify Euclids proof of the infinitude of prime numbers.

Problem 2. As discussed in class, Pythagoras and his follower are credited with the study of perfect numbers. In Euclid's *Elements* the following result is proved:

Theorem 1. *Let n be a positive integer such that $2^{n+1}-1$ is a prime number. Then the number $2^n(2^{n+1}-1)$ is perfect.*

a) Prove this theorem (Hint: what are the proper divisors of $2^n p$ when p is a prime number ? what is their sum ?).

b) Prove conversely, that if a number of the form $2^n(2^{n+1}-1)$ is perfect then $2^{n+1}-1$ is a prime number.

c) Prove that if 2^m-1 is a prime then so is m (Hint: do a proof by contrapositive, i.e. show that if m is composite then so is 2^m-1).

d) (Extra credit) Prove the result of Euler, that if N is an even perfect number then $N = 2^n p$, where $p = 2^{n+1}-1$ is a prime number.

Problem 3. a) Read Proposition 5 in Book 2 of the Elements. Explain how this proposition implies the inequality $(a+b)/2 \geq \sqrt{ab}$.

b) Read Proposition 2 and its proof in Book 12 of the Elements. Then write the proof in your own words and explain the use of the method of exhaustion.

c) Proposition 11 in Book 4 of the Elements provides a construction of a regular pentagon inscribed in a given cricle. Write this construction in your own words and justify that it is correct.

Problem 4. Read the Remark at the end of Book 13 of the Elements. It contains a proof that there are only 5 Platonic solids. Explain the idea of this proof. See the links provided on the course web page for more on Platonic solids.

Problem 5. a) What number is represented by the continued fraction $[1, 2, 3, 1, 2, 3, 1, 2, 3, ...]$? Justify your answer.

b) Find a continued fraction expansion of 104348/33215. Do not use calculators.

c) Find the continued fraction expansion of $\sqrt{7}$. Justify your answer.