Homework 4
due on Thursday, April 7

Read carefully chapters 4, 5 of Dunham's book. Also solve the following problems.

Problem 1. Let $ABCD$ be a square of side $2R$. Let O be the center of the square and let l be the line parallel to BC passing through O. Consider the solid obtained by revolving triangle DOA about the line l. Also consider the sphere with center O and radius R. Prove that the sections of the sphere and the solid by any plane perpendicular to the line l have equal areas. Using this explain how Archimedes would use his "Method" to argue that the solid and the sphere have equal volumes. Compute the volume of the solid and derive a formula for the volume of a sphere.

Problem 2. In his "quadrature of a parabola" Archimedes used the following two "well known" facts about parabolas. Let A, B be two points on a parabola and let l be the line parallel to the axis of the parabola and passing through the midpoint M of AB. Let P be the point where l meets the parabola. Then

1. The tangent to the parabola at P is parallel to the line AB.
2. The tangent to the parabola at B intersects l at a point N such that $NP = PM$.

a) Assuming that the parabola is given by the equation $y = x^2$, use analytic geometry (i.e. coordinates) to prove both facts. Hint: The tangent line to a curve $y = f(x)$ at a point $(a, f(a))$ has slope equal to $f'(a)$.

b) Again, assuming that the parabola is given by $y = x^2$, use integration to prove that the area of the segment of the parabola cut by AB is equal to 4/3 times the area of the triangle ABP.

Remark: There is no loss of generality in the assumption above: it has been already known to the Greeks that any two parabolas are similar.

Problem 3. a) Let $ABCD$ be a trapezium such that $AB \parallel CD$, $AB \perp BC$, and $AB \geq CD$. Let M, N be the midpoints of AD and BC respectively. Let K be the point on the line AB such that $DK \perp AB$ and let the line perpendicular to AD at M intersect BC at L. When the trapezium is revolved about the line BC we get a frustum of a cone. We proved in class that the surface area S of the frustum (without the top and bottom) is equal to $S = \pi AD(AB + CD)$. Prove that $S = 2\pi DK \cdot ML$ (hint: express MN in terms of AB and CD; compare triangles DKA and MNL).

b) Consider a regular $2n$-gon $A_1 \dots A_{2n}$. Prove that the solid obtained by revolving the polygon about the perpendicular bisector of the segment $A_1 A_{2n}$ has surface area S equal to $4\pi r^2 + \pi a^2/2$, where r is the radius of the circle inscribed in the polygon and $a = A_1 A_2$ is the side length of the polygon.

c) Continuing part b), let R be the radius of the circle circumscribed on the polygon. Prove that $r^2 = R^2 - a^2/4$ and conclude that $S = 4\pi R^2 - \pi a^2/2$.

Problem 4. Consider a deltoid $ABCD$ (this means that the triangles ABC and ADC are congruent) with $AB = AD < CB = CD$. Let M, N be points on the segments CB, CD respectively such that $BM = DN$. The goal of this problem is to compute the volume V of the solid obtained by revolving triangle ABM about the line AC. Let P be the point on the line BC such that $PA \perp BC$.

a) Prove that $V = \pi(BD^2 - MN^2)AC/12$.

b) Prove that $AC(DB - MN)/2 = PA \cdot BM$ (compare triangles APC and BKM, where K is the point on BD such that $MK \perp BD$). Conclude that $V = S \cdot PA/3$, where S is the surface area of the frustum of a cone obtained by revolving segment BM about line AC (without the top and bottom).

Problem 5. Consider a circle of radius R and two regular $2n$-gons, $B_1 \dots B_{2n}$ and $A_1 \dots A_{2n}$, one inscribed and the other circumscribed on the circle. We assume that the lines $A_1 A_2$ and $B_1 B_2$ are parallel. Revolving the circle and the polygons about the perpendicular bisector of $A_1 A_{2n}$ (which coincides with the bisector of $B_1 B_{2n}$) we get a sphere and two solids T_{out} and T_{in}, the former containing the sphere and the latter contained inside it. Let S_{out}, S_{in} be the surface area of T_{out}, T_{in} respectively and let V_{out}, V_{in} be their volumes. Let $a = A_1 A_2$, $b = B_1 B_2$.

a) Use Problem 3 to prove that $S_{out} = 4\pi R^2 + \pi a^2/2$ and $S_{in} = 4\pi R^2 - \pi b^2/2$. Explain how to conclude that the surface area of the sphere is $4\pi R^2$.

b) Use the formula from Problem 4 b) to show that the volumes of T_{out}, T_{in} are equal to $S_{out}R/3$ and $S_{in}Rb/3a$. Explain how to conclude that the volume of the sphere is $4\pi R^3/3$

Problem 6. Consider a circle with center O and let A, B be two points on the circle which are not ends of a diameter. Extend the ray $AO^{\rightarrow}$ and find on it a point D (outside the circle) such that the line BD intersects the circle at a point E and $ED = BO$. Prove that the angle ADB is one third of the angle AOB. (This is a result from the "Book of Lemmas" by Archimedes. It is Archimedes' contribution to the problem of angle trisection; it is similar in spirit to the angle trisection via conchoid).