Homework

due on Monday, May 9

Read carefully chapters 7, 8, 9, 10 of Dunham's book. Solve the following problems.

Problem 1. Consider a cubic equation $x^3 + px + q = 0$.

a) Show that if $p \geq 0$ then this equation has unique real solution (use calculus).

b) Suppose now that $p < 0$. Prove that the function $f(x) = x^3 + px + q$ has a local maximum at $x_1 = -\sqrt{-p/3}$ and a local minimum at $x_2 = \sqrt{-p/3}$. Conclude that $f(x) = 0$ has more than one real solution iff $f(x_1) \geq 0$ and $f(x_2) \leq 0$.

c) Use b) to show that if $p < 0$ then $x^3 + px + q = 0$ has more than one real solution iff

$$\Delta = \frac{p^3}{27} + \frac{q^2}{4} \leq 0$$

and it has three distinct solutions iff $\Delta < 0$.

Problem 2. This problem describes Viete's approach to cubic equations which have three real roots. Let $x^3 + px + q = 0$ be such an equation. We know from the previous problem that $p < 0$. Let $R = \sqrt{-p/3}$.

a) Use b) or c) of the previous problem to show that

$$-1 < \frac{q}{2R^3} < 1.$$

Conclude that

$$\frac{-q}{2R^3} = \cos\phi$$

for some $\phi \in (0, \pi)$.

b) Prove that $\cos(3a) = 4\cos^3(a) - 3\cos(a)$ for every a (use the well known formulas for $\cos(a + b)$, $\cos(2a)$ and $\sin(2a)$).

c) For $i = 0, 1, 2$ let

$$\phi_i = \frac{\phi}{3} + i\frac{2\pi}{3},$$

where ϕ is defined in part a). Prove that $x_i = 2R\cos\phi_i$ is a root of $x^3 + px + q = 0$ for $i = 0, 1, 2$ and that these roots are disctinct.

d) Show that $x_1 = -R[\cos(\phi/3) + \sqrt{3}\sin(\phi/3)]$ and $x_2 = -R[\cos(\phi/3) - \sqrt{3}\sin(\phi/3)]$.

e) Use Viete's method to solve $x^3 - 9x + 3\sqrt{6} = 0$. Then use Cardano's formula to solve it. What can you say?

Problem 3. a) Find all 4 roots of the equation

$$x^4 - 6x^2 - 24x - 3.$$

Recall: the idea is to find u, w, t such that the equation can be written as

$$(x^2 + u)^2 = t(x + w)^2.$$

b) Find all roots of the equation

$$x^4 + 4x^3 - 32x - 32.$$

Hint: Start with a substitution which eliminates the x^3 term.