Midterm, Math 498/Math 575P October 28, 2013

In order to get full credit a correct solution must be written carefully, with detailed explanation of all steps. Each problem is worth 10 points.

Problem 1. Tribonacci numbers T_n are defined as follows: $T_1 = T_2 = T_3 = 1$, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \ge 4$. Prove by induction that $T_n < 2^n$ for every natural number n.

Solve three of the following 4 problems.

Problem 2. Find the value of the integral $\int_{-\pi/2}^{\pi/2} \frac{\cos x dx}{2^x + 1}$.

Problem 3. Find all positive integers n such that $n^2 + 6n$ is a square of an integer. Prove your answer.

Problem 4. 10 numbers are selected from the set $\{1, 2, ..., 23\}$. Prove that among the selected numbers there are two disjoint pairs of numbers with the same sum. Hint: A set with *m* elements has $\binom{m}{2}$ 2-element subsets.

Problem 5. Numbers 1, 2, 3, ..., 2014 are written on a blackboard. Every now and then somebody picks three numbers a, b, c and replaces them by a + b - c, b + c - a, a + c - b. Is it possible that at some point all 2014 numbers on the blackboard are equal? Hint: 1 + 2 + ... + n = n(n + 1)/2.