Solutions to the Midterm, Math 498/Math 575P

Problem 1. Tribonacci numbers T_n are defined as follows: $T_1 = T_2 = T_3 = 1$, $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for all $n \geq 4$. Prove by induction that $T_n < 2^n$ for every natural number n.

Solution. First we verify that the inequality is true for $n = 1, 2, 3$:

$$T_1 = 1 < 2^1, \quad T_2 = 1 < 2^2, \quad T_3 = 1 < 2^3.$$

Suppose now that $n \geq 4$ and that $T_k < 2^k$ for $k = 1, \ldots, n-1$. Then $T_{n-1} < 2^{n-1}$, $T_{n-2} < 2^{n-2}$, $T_{n-3} < 2^{n-3}$. Thus

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} < 2^{n-1} + 2^{n-2} + 2^{n-3} = 2^n \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \right) = 2^n \cdot \frac{7}{8} < 2^n$$

so the ineuqlity holds also for n. By the method of mathematical induction, the inequality $T_n < 2^n$ is true for every natural number n.

Problem 2. Find the value of the integral $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x \; dx}{2^x + 1}$.

Solution. We use substitution $x = -y$, $dx = -dy$ to get

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x \; dx}{2^x + 1} = \int_{\pi/2}^{-\pi/2} \frac{\cos(-y)(-dy)}{2^{-y} + 1} = \int_{-\pi/2}^{\pi/2} \frac{\cos y \; dy}{\frac{1}{2^y} + 1} = \int_{-\pi/2}^{\pi/2} \frac{2^y \cos y \; dy}{1 + 2^y} = \int_{-\pi/2}^{\pi/2} \frac{2^x \cos x \; dx}{2^x + 1}.$$

It follows that

$$2 \int_{-\pi/2}^{\pi/2} \frac{\cos x \; dx}{2^x + 1} = \int_{-\pi/2}^{\pi/2} \frac{\cos x \; dx}{2^x + 1} + \int_{-\pi/2}^{\pi/2} \frac{2^x \cos x \; dx}{2^x + 1} =$$

$$= \int_{-\pi/2}^{\pi/2} \frac{(\cos x + 2^x \cos x) \; dx}{2^x + 1} = \int_{-\pi/2}^{\pi/2} \cos x \; dx = 2,$$

so

$$\int_{-\pi/2}^{\pi/2} \frac{\cos x \; dx}{2^x + 1} = 1.$$

Problem 3. Find all positive integers n such that $n^2 + 6n$ is a square of an integer. Prove your answer.

Solution. Note that

$$n^2 + 6n < n^2 + 6n + 9 = (n+3)^2 \quad \text{and} \quad n^2 + 6n > n^2 + 2n + 1 = (n+1)^2$$

for every natural number n. Thus

$$(n+1)^2 < n^2 + 6n < (n+3)^2$$

for every natural number n. It follows that $n^2 + 6n$ is a square if and only if $n^2 + 6n = (n+2)^2 = n^2 + 4n + 4$, which is equivalent to $n = 2$. Thus $n^2 + 6n$ is a square if and only if $n = 2$.

Problem 4. 10 numbers are selected from the set $\{1, 2, \ldots, 23\}$. Prove that among the selected numbers there are two disjoint pairs of numbers with the same sum. Hint: A set with m elements has $\binom{m}{2}$ 2-element subsets.

Solution. By the hint, there are $\binom{10}{2} = 45$ different pairs consiting of 2 of the selected numbers. The sum of the two numbers in any such pair is at lest $1 + 2 = 3$ and at most $23 + 22 = 45$. Thus the sum is one of the 43 numbers $3, 4, \ldots, 45$. We heve then 45 pairs (pigeons) which are assigned to 43 possible sums (holes). By the pigeon-hole principle, there are two different pairs a, b and c, d of the selected numbers such that $a + b = c + d$. Note that if a was equal to one of c, d then b would be equal to the other of c, d and the pairs would be the same. Thus these two pairs are disjoint.

Problem 5. Numbers $1, 2, 3, ..., 2014$ are written on a blackboard. Every now and then somebody picks three numbers a, b, c and replaces them by $a + b - c$, $b + c - a$, $a + c - b$. Is it possible that at some point all 2014 numbers on the blackboard are equal? Hint: $1 + 2 + \ldots + n = n(n+1)/2$.

Solution. Note first that at any time the numbers on the blackboard are all integers. Note also that the sum $a + b + c$ of the three erased numbers is the same as the sum $(a + b - c) + (b + c - a) + (a + c - b) = a + b + c$ of the new numbers. It follows that the sum of all the numbers on the blackboard is alsways the same and equal to $1 + 2 + \ldots + 2014 = 2014 \cdot 2015/2$. If at some point all the numbers were equal to a then their sum would be $2014 \cdot a$, so $a = 2015/2$, which is not an integer. Thus the numbers will never be all equal to each other.