Homework 2

due Monday, October 6

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Problem 1. Initially, we are given the sequence $1, 2, ...100$. Every minute, we erase two numbers u and v and replace them with the value $uv + u + v$. Clearly, we will be left with just one number after 99 minutes. What is this number? Justify your answer. (Hint:Look for an invariant.)

Problem 2. Start with the set $\{3, 4, 12\}$. You are then allowed to perform a sequence of replacements, each time replacing two numbers a and b from your set with the new pair $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you transform the set into $\{4, 6, 12\}$? Look for an invariant (i.e. a quantity which does not change).

Problem 3. Consider an $m \times n$ table of integers. If the sum of all elements in a row or column is negative you may change the sign of all numbers in that row (or column). Prove that at some point the sum of elements in every row and column will be nonnegative. Look for a monovariant (something which increases in each step and is bounded). Prove that the problem remains true if the numbers in the table are any real numbers.

Problem 4. Two rivers run parallel 2 miles appart. Two cities A and B lie between the rivers; each city is equidistant from the rivers and the cities are 3 miles appart. A scientist wishes to travel from A to B, collecting a sample of water from each river during his journey. What is the length of the shortest path he can follow. Justify your answer.

Problem 5. Compute $\displaystyle\int_0^2 \frac{x^3 dx}{6x^2 - 12x + 8}$. Hint: expand $(2 - x)^3$?