Homework 3

due Wednesday, October 22

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Problem 1. Let a, b be real numbers such that $a+b$ and ab are integers.

a) Prove that $a^n + b^n$ is an integer for every natural number n.

b) Suppose that $a \neq b$. Prove that $\frac{a^n - b^n}{a - b}$ is an integer for every positive integer n.

Problem 2. Prove that for every natural number $n > 1$ we have

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{2n-1}{2n} > \frac{1}{2\sqrt{n}}.$$

Problem 3. Prove that among any five points selected inside an equilateral triangle with side equal to 2, there always exists a pair at the distance not greater than 1.

Problem 4. 21 integers are selected from $\{1, 2, 3, ..., 400\}$. Prove that two of them, say x and y, satisfy $0 < |\sqrt{x} - \sqrt{y}| < 1$.

Problem 5. Nine distinct points with all coordinates integral are selected in the space. Prove that the line segment with ends at certain two of these points contains in its interior a point with all coordinates

Problem 6. Compute the integrals

$$\int_0^{\pi} \frac{x^2 \sin x}{x^2 + (\pi - x)^2} dx,$$

$$\int_0^{\pi} \frac{x^3 \sin x}{3x^2 - 3\pi x + \pi^2} dx.$$

Problem 7. Numbers $1, 2, 3, ..., 2014$ are written on a blackboard. Every now and then somebody picks two numbers a and b and replaces them by $2a - b$, $2b - a$. Is it possible that at some point all numbers on the blackboard are equal? Can they all be divisible by 3? Carefully justify your answer.