Homework 1
due on Wednesday, February 9

Problem 1. Let G be a group. Prove that:

a) If $a \in G$ has finite order n then, for any integer k, the order of a^k is $n/(n,k)$.

b) If a has order m, b has order n, and $ab = ba$ then the order of ab divides $mn/(m,n)$ and is divisible by $mn/(m,n)^2$.

c) If G has an element a of order m and an element b of order n such that $ab = ba$ then G has an element of order $[m,n]$ ($[m,n]$ is the least common multiple of m and n).

d) If G is a finite abelian group and N is the smallest positive integer such that $g^N = e$ for all $g \in G$, then G has an element of order N.

Remark. In general, the N defined in d) makes sense for any group (it can be infinite) and it is called **the exponent of** G.

e) If $f : G \longrightarrow H$ is a homomorphism and $a \in G$ has finite order n, then $f(a)$ has also finite order k which divides n. Also, a^m is in the kernel of f iff k divides m.

f) Let G be a cyclic group of order n and H a cyclic group of order m (we allow the orders to be infinite). Show that the set of all homomorphism from G to H is a group with operation $+$ defined by $(f+g)(a) = f(a)g(a)$ (this is true for arbitrary G and abelian H). Describe this group for each pair m, n.

Problem2. Let G be a group and H its subgroup.

a) Show that if a_iH, $i \in I$ are the left cosets of H in G then Ha_i^{-1}, $i \in I$ are the right cosets of H in G. Conclude that the number of left cosets of H is finite iff the number of right cosets is finite and these numbers coincide. The number of left (right) cosets of H in G is called the **index** of H in G and it is usually denoted by $[G:H]$.

b) Prove that if $K < H < G$ then $[G:K] = [G:H][H:K]$.

c) Show that for any subgroup K of G we have $[K : H \cap K] \leq [G:H]$.

d) Prove that if H, K are subgroups of G of finite index then so is $H \cap K$ and $[G : H \cap K] \leq [G:H][G:K]$.

e) Prove that if H is of finite index then G is finitely generated iff H is finitely generated.

f) Prove that if H is of finite index then there is a normal subgroup of G of finite index contained in H (show that the number of conjugates of H is finite and take their intersection).

g) Show that if G is finitely generated then it has only finitely many subgroups of a given finite index n (use the fact that the action of G on cosets of a subgroup K of index n defines a homomorphism of G into S_n whose kernel is contained in K).

h) If $[G:H] = 2$ then H is normal.

i) Show that if $[G:H] = n$ then $g^{n!} \in H$ for all $g \in G$. If H is normal then $n!$ can be replaced by n. Show that without normality this is no longer true.

Problem 3. a) Describe all subgroups and normal subgroups of D_n.

b) Describe the center and the derived group of D_n.

c) For which m, n is there a surjective homomorphism from D_m to D_n ? (**Optional:** Describe all homomorphisms from D_m to D_n.)

d) Show that $< a, b | a^2 = 1 = b^2 >$ is a presentation of D_∞.

e) Prove that if x, y are two elements of order 2 in a group G and $xy \neq yx$ then the subgroup $< x, y >$ of G is isomorphic to a dihedral group (finite or infinite).

Problem 4. a) Describe all subgroups of Q_8 and show that they are all normal.

b) Prove that the quaternion group Q_8 is isomorphic to the subgroup of $GL_2(\mathbb{C})$ generated by the matrices $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

c) Show that $< a, b | a^4 = 1, a^2 = b^2, aba = b >$ is a presentation of Q_8.

Furthermore, solve problems 18 , 23 to 1.6, problem 5 to 2.1, problem 26 to 2.3 and problem 1 to 6.3.

Challenge. Let n, m, k be positive integers, all greater than 1. Prove that there exists a finite group which contains elements a, b such that a has order n, b has order m and ab has order k.