Homework 4

due on Wednesday, March 30

Read carefully sections 4.5, 5.4, 5.5, 6.2 in the book.

Solve problems 16,22,51 to section 4.5, problems 3,8,10 to section 5.4, problems 6,7,8 to section 5.5, and the following problems.

Problem 1. Let $G = \sum_{i=-\infty}^{\infty} C$, where $C = \{0,1\}$ is the group of order 2. Thus G consists of all functions $f : \mathbb{Z} \longrightarrow C$ such that $f(i) = 0$ for all but finitely many i. Define an automorphism $t : G \longrightarrow G$ by $(tf)(i) = f(i-1)$ (so it is a shift). Let $\phi : \mathbb{Z} \longrightarrow \mathrm{Aut}G$ be given by $\phi(1) = t$. Set $H = G \rtimes \mathbb{Z}$.

a) Prove that G is not finitely generated and that H is generated by $(0,1)$ and $(g,0)$ where $g(0) = 1$ and $g(i) = 0$ for all $i \neq 0$.

b) Show that $[H,H]$ is the subgroup of G which consists of all f for which $f(i) = 1$ for an even number of i. Deduce that H,H is not finitely generated.

c) Conclude that the derived group of a free group on two generators is not finitely generated (this is true for any nonabelian free group).

Problem 2. Let G be a finite group and H a minimal nontrivial normal subgroup of G (i.e. nontrivial normal subgroup of G which does not contain any proper, nontrivial normal subgroup of G). Prove that H is a direct product of several copies of a simple group.

Problem 3. a) Prove that if $\phi, \psi : H \longrightarrow \mathrm{Aut}K$ are such that $\phi^{-1}\psi$ is an inner automorphism, then the groups $K \rtimes_\phi H$ and $K \rtimes_\psi H$ are isomorphic.

b) Suppose furthermore that there is no surjective homomorphism from K onto $\mathbb{Z}$ and that $H = \mathbb{Z}$. Show that $K \rtimes_\phi H$ and $K \rtimes_\psi H$ are isomorphic iff the images of ϕ^ϵ and ψ in the group $\mathrm{Out}K = \mathrm{Aut}K/\mathrm{Inn}K$ are conjugate in $\mathrm{Out}K$, where $\epsilon = 1$ or $\epsilon = -1$.

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Challenge. Let G be the direct product of countable many copies of $\mathbb{Z}$, i.e. $G = \Pi_{i=1}^{\infty} A_i$, where $A_i = \mathbb{Z}$ for all i. Let H be the direct sum of these groups.

a) Prove that if $\phi : G \longrightarrow \mathbb{Z}$ is a homomorphism such that $H < \ker\phi$, then $\ker\phi = G$.

b) Prove that G is not isomorphic to a direct sum of the form $\sum_{i\in I} \mathbb{Z}$.

c) An abelian group without elements of finite order B is called **slender**, if every homomorphisms $\psi : G \longrightarrow B$ maps all but a finite number of A_i to the identity of B. Prove that $\mathbb{Z}$ is slender.

d) Prove that there is no epimorphism of G onto H.