Homework 6

due on Monday, May 9

Read carefully sections 12.1, 12.2, 12.3 in the book.

Solve the following problems.

Problem 1. Let R be a PID. For an R-module M and a maximal ideal $J = (\pi)$ define $M_J = \{m \in M : \pi^k m = 0 \text{ for some } k > 0\}$.

a) Show that M_J is a submodule of M. It is called the J-primary component of M (if M is an abelian group and $J = p\mathbb{Z}$ with p a prime number then M_J is the Sylow p-subgroup of M).

b) Prove that the torsion submodule $T(M)$ is the direct sum of the J-primary components, where J runs through the maximal ideals of R (this may be an infinite direct sum, since we do not assume here that M is finitely generated). If M is finitely generated, show that M_J is non-zero for a finite number of maximal ideals J.

c) Recall that a finitely generated torsion R-module T is a direct sum of cyclic modules of the form $R/(\pi^k)$ for some prime element π and some k. Show that T_J is the direct sum of all factors for which $(\pi) = J$. Conclude that to show the uniqueness of the elementary divisors for T it suffices to show that for each maximal ideal J and each decomposition of T_J into a direct sum of cyclic modules the number $N(J, k)$ of direct summands isomorphic to R/J^k is independent of the decomposition.

d) Suppose that $T_J \approx R/J^{k_1} \oplus R/J^{k_2} \oplus ... \oplus R/J^{k_s}$ for some $k_1 \leq k_2 \leq ... \leq k_s$. Show that $JT_J \approx R/J^{k_1 - 1} \oplus R/J^{k_2 - 1} \oplus ... \oplus R/J^{k_s - 1}$ and $T_J/JT_J \approx R/J \oplus R/J \oplus ... \oplus R/J$ (s times). Note that T_J/JT_J is an R/J-module, i.e. a vector space over the field R/J and conclude that s is the dimension of this vector space.

e) Conclude from d) that for every natural number k the integer $S(k, J) := N(J, k) + N(J, k+1) + N(J, k+2) + ...$ is the dimension of the R/J-vector space $J^{k-1}T_J/J^kT_J$, hence it does not depend on the choice of the decomposition. Since $N(J, k) = S(J, k) - S(J, k+1)$, the uniqueness of the elementary divisors follows.

Problem 2. Let R be a PID. Suppose that M is a torsion R-module and $m \in M$ is such that $\text{ann}(m) = (r) = \text{ann}(M)$.

a) Show that in the set of all submodules of M which intersect $< m >$ trivially there is a maximal element N (with respect to inclusion). The next steps will show that M is the direct sum of $< m >$ and N. Show that this is equivalent to the statement that $M/N =< m + N >$ is a cyclic module.

b) Show that $\text{ann}(m + N) = (r) = \text{ann}(M/N)$ and for every non-zero element $x \in M/N$ the cyclic modules $< x >$ and $< m + N >$ have non-trivial intersection.

c) Suppose that M is a torsion R-module and $m \in M$ is such that $\text{ann}(m) = (r) = \text{ann}(M)$. For $0 \neq n \in M$ consider the set $I = \{a \in R : an \in < m >\}$. Show that it is an ideal of R which contains r. Let b be a generator of I, so $b|r$. Note that $bn = cm$ for some $c \in R$. Show that r divides $(r/b)c$ and conclude that $b|c$. Prove that $< m > \cap < n - (c/b)m >= \{0\}$.

d) Use c) to show that in b) we have $M/N =< m + N >$.

e) Show that if M is finitely generated then m with the required property exists (do not use the results about decomposition into a direct sum of cyclic modules).

This problem gives a different proof of the fact that a finitely generated torsion module is a direct sum of cyclic modules.

Problem 3. Find the rank, the invariant factors and the elementary divisors of the group $\mathbb{Z}^4/H$, where H is generated by $(-1,-2,-3,-4)$, $(3,8,5,6)$, $(-1,0,-13,-16)$, $(-3,-4,-13,-6)$. (Find a compatible bases of $\mathbb{Z}^4$ and H).