Solutions to the Midterm

Solution to Problem 1. Let H be a subgroup of a group G such that every element of H commutes with every element of $[G,G]$. For $g \in G$ consider the function $f_g : H \longrightarrow G$ defined by $f_g(h) = [h,g] = h^{-1}g^{-1}hg$. Note that

$$f_g(h_1h_2) = [h_1h_2, g] = h_2^{-1}h_1^{-1}g^{-1}h_1h_2g = h_2^{-1}[h_1,g]h_2[h_2,g] = h_2^{-1}f_g(h_1)h_2f_g(h_2)$$

Since $f_g(h_1)$ belongs to $[G,G]$, it commutes with all elements of H. In particular, $h_2^{-1}f_g(h_1)h_2 = f_g(h_1)$ and therefore

$$f_g(h_1h_2) = f_g(h_1)f_g(h_2)$$

which proves that f_g is a homomorphism.

In order to prove that the image of f_g is abelian we first make the following observation. If ϕ is an automorphism of G then every element of $\phi(H)$ commutes with every element of $\phi([G,G])$. Recall now that $[G,G]$ is a characteristic subgroup of G, so $\phi([G,G]) = [G,G]$. Thus every element in $\phi(H)$ commutes with all elements in $[G,G]$. We apply this to the inner automorphism $\phi(x) = g^{-1}xg$. Note that $f_g(h) = h^{-1}\phi(h)$ and since both h^{-1} and $\phi(h)$ commute with all elements in $[G,G]$, so does $f_g(h)$. If h_1 is another element of H then $f_g(h_1) \in [G,G]$ and therefore $f_g(h)$ and $f_g(h_1)$ commute. This shows that the image of f_g is an abelian group.

It follows that $[H,H] \subseteq \ker f_g$, i.e g commutes with all elements in $[H,H]$. Since this holds for every $g \in G$, we see that $[H,H]$ is contained in the center of G.

Solution to Problem 2. Let f be an automorphism of a finite group G such that $f(g) = g^{-1}$ for some nonidentity element g of order larger than 2. Let k be the order of f, so f^k is the identity. In particular, $g = f^k(g) = f(f(f...(f(g))...)) = g^{(-1)^k}$. Since the order of g is larger than 2, we have $g \neq g^{-1}$. It follows that $(-1)^k = 1$, i.e. k is even.

Suppose now that g and g^{-1} are conjugate so $g^{-1} = aga^{-1}$ for some $a \in G$. Consider the inner automorphism $f(x) = axa^{-1}$. By the first part of our solution, f must have even order. But the order of f divides the order of a (recall that f is the image of a under the natural homomorphism $G \longrightarrow \mathrm{Inn}G$), so a has even order contrary to our assumption that the order of G is odd. Thus g and g^{-1} cannot be conjugate.

Solution to Problem 3. Let A, B be finite index subgroups of a group G. By the very definition,

$$AB = \bigcup_{a\in A} aB.$$

Note that for $a_1, a_2 \in A$ we have $a_1B = a_2B$ iff $a_2^{-1}a_1 \in B$ i.e. iff $a_2^{-1}a_1 \in A \cap B$. It follows that among the left cosets aB with $a \in A$ there are exactly $[A : A \cap B]$ different cosets. Thus AB is a union of $[A : A \cap B]$ different left cosets of B in G. Since AB is a subset of G and G is the union of $[G : B]$ distinct left cosets of B in G, we see that $[A : A \cap B] \leq [G : B]$ and the equality holds iff $G = AB$.

Suppose now that $[G : A]$ and $[G : B]$ are relatively prime. Note that

$$[G : A][A : A \cap B] = [G : A \cap B] = [G : B][B : A \cap B].$$

In particular, $[G : B]|[G : A][A : A \cap B]$. Since $[G : A]$ and $[G : B]$ are relatively prime, we conclude that $[G : B]|[A : A\cap B]$. Consequently, $[G : B] \leq [A : A\cap B]$. As we observed above, the converse inequality is always true, so $[A : A \cap B] = [G : B]$ and $G = AB$.

Solution to Problem 4. Let G be a group of order 525. We will show that G has a normal Sylow 5-subgroup. The number t_5 of Sylow 5-subgroups of G divides 21 and is congruent to 1 modulo 5. It follows that $t_5 = 1$ or $t_5 = 21$. Suppose that $t_5 = 21$. Let P, Q be distinct Sylow 5-subgroups. If $P \cap Q = \{1\}$ then the set PQ has $25^2 > 525$ elements, which is not possible. Thus any two Sylow 5-subgroups have non-trivial intersection. Let $R = P \cap Q$. Since both P and Q are abelian, the centralizer $C_G(R)$ contains both P and Q. Thus P and Q are distinct Sylow 5-subgroups of $C_G(P)$. Furthermore, $C_G(R)$ has order $3 \cdot 25$, $7 \cdot 25$ or 525. But in the first two cases Sylow's theorem implies that $C_G(R)$ has unique (i.e normal) Sylow 5-subgroup, which is not possible. Thus only the third case is possible, i.e $C_G(R) = G$. Thus R is normal (even central) in G. Since all Sylow 5-subgroups are conjugate, they all contain R. It follows that R is the intersection of <u>any</u> two distinct Sylow 5-subgroups of G. Hence the number of elements of order 5 or 25 in G is $4 + 21 \cdot 20 = 424$.

We show now that both $t_7 > 1$ and $t_3 > 1$, i.e. Sylow 3-subgroups and Sylow 7-subgroups are not normal. In fact, suppose that N is a normal Sylow q-subgroup of G with $q = 3$ or $q = 7$. Any Sylow 5-subgroup P of G acts on N by conjugation. But $5 \nmid (q-1)$ (which is the order of the group of automorphisms of N), so the action is trivial, i.e. P centralizes N. Thus N is contained in the normalizer (even centralizer) of P. This is however not possible, since $t_5 = 21$ implies that the normalizer of P has index 21 i.e. is equal to P. The contradiction shows that both $t_7 > 1$ and $t_3 > 1$. Sylow Theorem implies now that $t_7 = 15$ and $t_3 \geq 7$. Thus G has $6 \cdot 15 = 90$ elements of order 7 and at least $2 \cdot 7 = 14$ elements of order 3. This means that G has at least $424 + 90 + 14 + 1 = 529$ elements, a contradiction. This shows that our assumption that $t_5 = 21$ is wrong, i.e $t_5 = 1$.

Remark. There are several other ways of proving that $t_5 = 1$. For example, after showing that R is normal, we could look at the factor group $H = G/R$ of order 105. It is not hard to show that a group of order 105 has a normal Sylow 5-subgroup and this implies that G has a normal Sylow 5-subgroup (by correspondence theorem).

We see that G has a normal subgroup P of order 25. Clearly P is solvable (even abelian). The group $K = G/P$ has order 21. Thus K has a normal Sylow 7-subgroup B and K/B has order 3. Thus both B and K/B are cyclic, hence solvable. It follows that $K = G/P$ is solvable and since P is solvable, so is G.

Remark. To show that G is solvable after proving only that G is not simple (as suggested in the problem) start with a normal subgroup H of G and show that both H and G/H are solvable. There are several cases to consider (the order of H can be 3,5,7, ...0) , but they are all fairly simple.