**NORMAL AND SUBNORMAL SERIES**

Let $G$ be a group. A **subnormal series** of subgroups of $G$ is a sequence $G_0 = \{e\} < G_1 < ... < G_n = G$ such that $G_{i-1} \triangleleft G_i$ for $i = 1, ..., n$. A **normal series** is a subnormal series such that each $G_i$ is normal in $G$. The number $n$ is called the length of the (sub)normal series. A (sub)normal series is called **proper** if all the quotient groups $G_i/G_{i-1}$ are nontrivial. These quotient groups are refered to as **successive quotients** of the (sub)normal series.

We say that two (sub)normal series are equivalent if they have the same length and there is a permutation $\pi \in S_n$ such that the groups $G_i/G_{i-1}$ and $G_{\pi(i)}/G_{\pi(i)-1}$ are isomorphic for all $i = 1, 2, ..., n$.

A **refinement** of a (sub)normal series $G_0 = \{e\} < G_1 < ... < G_n = G$ is a (sub)normal series $G'_0 = \{e\} < G'_1 < ... < G'_m = G$ such that there is a strictly increasing function $f : \{1, ..., n\} \longrightarrow \{1, 2, ..., m\}$ such that $G_i = G'_{f(i)}$ for all $i$.

We prove now the following important theorem:

**Schreier Refinement Theorem.** *Any two (sub)normal series in $G$ have equivalent refinements. It the series are proper, the refinements can be chosen proper as well.*

*Proof:* Note first that each integer $k$ can be uniquely written in the form $(m+1)i+j$ for some integer $i$ and some $j \in \{0, ..., m\}$. If $0 \le k < (m+1)(n+1)$ then $0 \le i \le n$. Similarly, each integer $0 \le k < (m+1)(n+1)$ can be uniquely written in the form $(n+1)j + i$, where $0 \le i \le n$ and $0 \le j \le m$.

For $1 \le k < (m+1)(n+1)$ write $k = (m+1)i+j$ as above and set $f(k) = (n+1)j+i$. It is clear that $f$ is a bijection of the set $\{1, ..., mn + m + n\}$.

Let $G_0 = \{e\} < G_1 < ... < G_n = G$ and $G'_0 = \{e\} < G'_1 < ... < G'_m = G$ be (sub)normal series.

Define $H_{(m+1)i+j} = (G_i \cap G'_j)G_{i-1}$ for $i = 0, ..., n$ and $j = 0, ..., m$ (the groups with negative indexes are by definition trivial).

Similarly, define $H'_{(n+1)j+i} = (G'_j \cap G_i)G'_{j-1}$ for $i = 0, ..., n$ and $j = 0, ..., m$.

It is clear that $H_0 = \{e\} < H_1 < ... < H_{mn+m+n} = G$ and $H'_0 = \{e\} < H'_1 < ... < H'_{mn+m+n} = G$ are (sub)normal series which are refinements of $G_0 = \{e\} < G_1 < ... < G_n = G$ and $G'_0 = \{e\} < G'_1 < ... < G'_m = G$ respectively. To see this note that $H_{(m+1)i+m} = G_i$ and $H'_{(n+1)j+n} = G'_j$. We claim that these series are equivalent. More precisely, the groups $H_k/H_{k-1}$ and $H'_{f(k)}/H'_{f(k)-1}$ are isomorphic for $k = 1, ..., mn + m + n$. Indeed, let $k = (m + 1)i + j$. If $j \neq 0$ then

$$H_k/H_{k-1} = (G_i \cap G'_j)G_{i-1}/(G_i \cap G'_{j-1})G_{i-1}$$

We apply the second homomorphism theorem to the group $A = G_i$, its subgroups $M = (G_i \cap G'_{j-1}) \triangleleft (G_i \cap G'_j) = N$ and its normal subgroup $G_{i-1} = B$. Recall that the second homomorphism theorem says that $NB/MB$ and $N/M(N \cap B)$ are isomorphic. In our case this means that

$$H_k/H_{k-1} \approx (G_i \cap G'_j)/(G_i \cap G'_{j-1})(G_{i-1} \cap G'_j).$$

If $j = 0$ the above formula still holds since in this case both sides are trivial groups.

Note now that $f(k) = (n+1)j+i$ and exactly the same argument as above shows that

$$H'_{f(k)}/H'_{f(k)-1} \approx (G'_j \cap G_i)/(G'_j \cap G_{i-1})(G'_{j-1} \cap G_i).$$

Since both $G'_j \cap G_{i-1}$ and $G'_{j-1} \cap G_i$ are normal subgroups of $G'_j \cap G_i = G_i \cap G'_j$, we have $(G'_j \cap G_{i-1})(G'_{j-1} \cap G_i) = (G_i \cap G'_{j-1})(G_{i-1} \cap G'_j)$. It follows that the groups $H_k/H_{k-1}$ and $H'_{f(k)}/H'_{f(k)-1}$ are indeed isomorphic.

This shows that $H_0 = \{e\} < H_1 < ... < H_{mn+m+n} = G$ and $H'_0 = \{e\} < H'_1 < ... < H'_{mn+m+n} = G$ are equivalent refinements of $G_0 = \{e\} < G_1 < ... < G_n = G$ and $G'_0 = \{e\} < G'_1 < ... < G'_m = G$ respectively. But this refinements are usually not proper even if the original series were proper. To end the proof it remains to make a rather obvious remark that omitting all repetitions in a (sub)normal series produces a proper (sub)normal series. Moreover

- this procedure applied to a refinement of a proper series results in another refinement which is proper;
- this procedure applied to equivalent series results in equivalent series. $\square$

A subnormal series is called a **composition series** if it is proper and does not have any nontrivial proper refinements. In other words, a composition series is a subnormal series $G_0 = \{e\} < ... < G_n = G$ such that $G_i/G_{i-1}$ is a nontrivial simple group for all $i$.

As an immediate consequence of the Schreier Refinement Theorem we get the following important result:

**Jordan-Hölder Theorem.** *Any two composition series of a group $G$ are equivalent.*

*Proof:* By the Schreier Refinement Theorem, the two composition series have equivalent proper refinements. Since composition series by definition do not have any nontrivial proper refinements, the result follows. □

As a consequence we see that the groups which appear as quotients of consequtive members of a composition series do not depend on the particular composition series. These groups are called **composition factors** of $G$. Also, the length of a series of $G$ is independent on the composition series and it is called the **composition length** of $G$.

Not all groups have composition series. For example, $\mathbb{Z}$ has no composition series since every nontrivial subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$, hence it is not simple (and in any composition series $G_1$ is simple).

On the other hand, we have the following

**Theorem 1.** *a) Finite groups have composition series.*

*b) Let $G$ be a group and $H$ its normal subgroup. Then $G$ has composition series iff both $H$ and $G/H$ have composition series. Moreover, the composition length of $G$ is the sum of composition lengths of $H$ and $G/H$.*

*Proof:* a) Use induction on the order of $G$. $G$ has a maximal proper normal subgroup $H$ and then $G/H$ is simple. If $G_0 = \{e\} < ... < G_n = H$ is a composition series for $H$ then $G_0 < ... < G_n < G_{n+1} = G$ is a composition series for $G$. (Alternatively, note that any proper subnormal series of $G$ has length bounded above by $|G|$, so there is a subnormal series with no proper nontrivial refinements.).

b) Exercise. □

**Exercise.** Prove that the group $GL_n(F)$ has composition series iff $F$ is finite.

**Exercise.** Prove that an abelian group has composition series iff it is finite.

**Exercise.** Find all composition series of $S_n$.

**Exercise.** Let $F$ be a finite field with $p$ elements, $p-$a prime. Find a composition series for the group $UT_n(F)$ of unipotent upper-triangular matrices.

**Exercise.** A **chief** series of $G$ is a normal series which is proper and does not have any proper nontrivial refinement (which is a normal series).

a) Show that any two chief series of $G$ are equivalent.

b) Prove that if $G$ has a composition series then it has a chief series. Is the converse true?

c) A **chief factor** of $G$ is a group isomorphic to the quotient of two consecutive members of a chief series of $G$. Show that if $G$ has a composition series then any chief factor of $G$ has is a direct product of some finite number of copies of a simple group.

## 1. Solvable and nilpotent groups

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Let $G$ be a group. We define a sequence of subgroups of $G$ inductively as follows: $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$, i.e. each term of this sequence is the derived group of the previous term. This sequence is called the **derived** series of $G$. It is clear that $G^{(i)}/G^{(i+1)}$ is abelian for all $i$.

A group $G$ is called **solvable** if $G^{(i)} = \{e\}$ for some $i$. The smallest $i$ for which this happens is called the **solvability class** of $G$.

Solvable groups have many normal subgroups so they are in a sense opposite to simple groups. A simple group is solvable iff it is abelian (hence cyclic of prime order). The class of solvable groups is very important. We have the following characterization of solvable groups.

**Theorem 2.** *The following conditions are equivalent:*

(1) *$G$ is solvable;*
(2) *$G$ has a normal series in which all successive quotients are abelian;*
(3) *$G$ has a subnormal series in which every successive quotient is abelian.*

*Proof:* The implications (1) $\Rightarrow$ (2) follows from the obvious fact that if $G$ is solvable then the derived series is a normal series. That (2) $\Rightarrow$ (3) is clear.

If $G_0 < ... < G_n$ is a subnormal series with abelian successive quotients then we show by induction that $G^{(i)} < G_{n-i}$, so in particular $G^{(n)} = \{e\}$. In fact, this is clear for $i = 0$ and if it holds for $i$, then $G^{(i+1)} = [G^{(i)}, G^{(i)}] < [G_{n-i}, G_{n-i}] < G_{n-i-1}$, the last inclusion being a consequence of the fact that $G_{n-i}/G_{n-i-1}$ is abelian. $\square$

**Theorem 3.** *a) Any subgroup of a solvable group is solvable.*

*b) If $G$ is solvable and $H \triangleleft G$ then $G/H$ is solvable.*

*c) If $H \triangleleft G$ and both $H$, $G/H$ are solvable then $G$ is solvable.*

*Proof:* a) This follows from the fact that if $H < G$ then $H^{(i)} < G^{(i)}$ for all $i$.

For b) note that under the projection $G \longrightarrow G/H$ the group $G^{(i)}$ is mapped onto $(G/H)^{(n)}$.

To show c) let $G_0 < ... < G_k = H$ and $B_0 < ... < B_l = G/H$ be a subnormal series with abelian successive quotients of $H$ and $G/H$ respectively. By the correspondence theorem we have $B_i = G_{k+i}/H$ for some subgroup $G_{k+i}$ of $G$. By the third homomorphism theorem, the group $G_i/G_{i-1}$ is isomorphic to $B_i/B_{i-1}$, hence it is abelian. Thus $G_0 < ... < G_{k+l}$ is a subnormal series of $G$ with abelian successive quotients. In particular, $G$ is solvable. $\square$

**Exercise.** Show that a solvable group has a composition series iff it is finite. Show that a finite group is solvable iff all its composition factors are cyclic of prime order. Show that each chief factor of a finite solvable group is a product of several copies of a cyclic group of prime order.

Besides derived series there are two other series associated to a group $G$. The **lower central series** for $G$ is defined inductively as follows: $G^{[0]} = G$ and $G^{[i+1]} = [G, G^{[i]}]$. Directly from the definition we see that $G^{[i]}$ are normal in $G$ and $G^{[i-1]}/G^{[i]}$ is contained in the center of $G/G^{[i]}$. The **upper central series** for $G$ is defined inductively as follows: $Z_0 = \{e\}$ and $Z_i/Z_{i-1}$ is the center of $G/Z_{i-1}$. It is easy to see that the groups in the upper central series are normal in $G$.

We say that a group $G$ is **nilpotent** if it has a normals series $G_0 < ... < G_n$ such that $G_{i+1}/G_i$ is contained in the center of $G/G_i$ for all $i$. Any such normal series is called a **central series** of $G$. The successive quotients of a central series are abelian so nilpotent groups are solvable, but the converse is false. For example, $S_3$ is solvable but not nilpotent.

It turns out that the following is true:

**Theorem 4.** *The following conditions are equivalent:*

(1) *$G$ is nilpotent;*
(2) *$G^{[i]} = \{e\}$ for some $i$.*
(3) *$Z_i = G$ for some $i$.*

*Moreover, if $G$ is nilpotent then the smallest $i$ such that $G^{[i]} = \{e\}$ coincides with the smallest $i$ such that $Z_i = G$. This number is called the* **nilpotency class** *of $G$.*

*Proof:* Let $G_0 < ... < G_m$ be a central series for $G$. Easy induction shows that $G^{[i]} < G_{m-i}$ for all $i$ and $G_i < Z_i$ for all $i$. In fact, the condition that $G_{i+1}/G_i$ is contained in the center of $G/G_i$ is equivalent to $[G, G_{i+1}] < G_i$. Thus, if $G^{[i]} < G_{m-i}$ then $G^{[i+1]} = [G, G^{[i]}] < [G, G_{m-i}] < G_{m-i-1}$. Also, if $G_i < Z_i$ then $G/Z_i$ is a quotient of $G/G_i$. In particular, the image of $G_{i+1}$ in $G/Z_i$ is central, so $G_{i+1} < Z_{i+1}$ by the definition of $Z_{i+1}$.

We see that if $G$ is nilpotent that the smallest $i$ such that $G^{[i]} = \{e\}$ is bounded above by the length of any central series in $G$. Consequently, the lower central series is a central series if $G$ and the smallest $i$ such that $G^{[i]} = \{e\}$ is in fact equal to the length of the shortest possible central series for $G$.

The same is true for the smallest $i$ such that $Z_i = G$. Thus we see that (1) implies both (2) and (3) and that our definition of nilpotence class is correct.

Conversely, both (2) and (3) imply (1) since the lower (upper) central series becomes a central series of $G$. $\square$

**Exercise.** Prove that:

a) Any subgroup of a nilpotent group is nilpotent.

b) If $G$ is nilpotent and $H \triangleleft G$ then $G/H$ is nilpotent.

**Exercise.** A subgroup $H$ of a group $G$ is called **characteristic** if $f(H) = H$ for every automorphism $f$ of $G$. Prove that charcteristic subgroup is normal. Prove that each memeber of a derived series, lower central series or upper central series is characteristic.

**Exercise.** A group $G$ is called **supersolvable** if it has a normal series with cyclic successive quotients.

a) Prove that finite nilpotent groups are supersolvable.

b) Prove that a finite group is supersolvable if all chief factors of $G$ have prime order.

**Exercise.** Show that the group $UT_n(F)$ is nilpotent and find its nilpotence degree.