Homework 2

due on Monday, March 4

Read carefully Chapter 2 in Milne's book. Solve problems 2-3, 2-7, 2-9. Read carefully sections 5.1, 5.2, 6.3 in Dummit & Foote. Solve problems 3, 8 to section 5.4 and problem 14 to section 5.2.

Solve the following problems.

Problem 1. Let H be a subgroup of $\mathbb{Z}^4$ generated by $(-1,-2,-3,-4)$, $(3,8,5,6)$, $(-1,0,-13,-16)$, $(-3,-4,-13,-6)$. Find a compatible bases of $\mathbb{Z}^4$ and H, then find the rank, the invariant factors, and the elementary divisors of the group $\mathbb{Z}^4/H$.

Problem 2. Suppose that M is a torsion abelian group of finite exponent t. Using the results of Problem 1 in Homework 1, M conatins an element m of order t (explain why).

a) Let $0 \neq n \in M$ and let s be the order of $n+ < m >$ in the quotient group $M/ < m >$. Note that this implies that $sn = rm$ for some r. Prove that $s|r$. Prove that $< m > \cap < n-(r/s)m >= \{0\}$.

b) Show that in the set of all subgroups of M which intersect $< m >$ trivially there is a maximal element N (with respect to inclusion).

c) Show that M is the direct sum of $< m >$ and N if and only if M/N is cyclic.

d) Show that M/N has exponent t and for every non-zero element $x \in M/N$ the cyclic subgroups $< x >$ and $< m+N >$ have non-trivial intersection.

e) Use d) and a) to show that $M/N =< m+N >$. Then use b) to conclude that $M =< m > \oplus N$.

f) Use e) to prove that a finite abelian group is a direct sum of cyclic groups (hence we get a different proof than the one in class).

Remark. As we mentioned in class, every abelian group of finite expenent is a direct sum of cyclic groups, but it is a bit harder to prove.

Problem 3. Let n be a positive integer. An element a in an abelian group A is called n-divisible if $a = nb$ for some $b \in A$. Let p be a prime. We say that a has infinite p-height if a is divisible by every power of p.

Consider the group $P = \prod_{i=1}^{\infty} < x_i >$, where x_i has order p^i. Let T be the torsion subgroup of P. Prove that no non-zero element of P has infinite height. Let $a = (a_i)$ be the element of P with $a_{2i} = x_{2i}^{p^i}$ and $a_{2i-1} = 0$ for $i = 1, 2, \ldots$. Prove that a is not in T and that $a+T$ has infinite p-height in P/T. Conclude that there is no subgroup B in P such that $P = T \oplus B$.

Problem 4. Let A, B, A_1, B_1 be groups such that the groups $A \times B$ and $A_1 \times B_1$ are isomorphic. Suppose that B and B_1 are isomorphic and finite. Prove that A and A_1 are isomorphic. Hint: One method is to do induction on the order of B.

Challenge. Let G be the direct product of countable many copies of $\mathbb{Z}$, i.e. $G = \Pi_{i=1}^{\infty} A_i$, where $A_i = \mathbb{Z}$ for all i. Let H be the direct sum of these groups.

a) Prove that if $\phi : G \longrightarrow \mathbb{Z}$ is a homomorphism such that $H < \ker \phi$, then $\ker \phi = G$.

b) Prove that G is not isomorphic to a direct sum of the form $\sum_{i \in I} \mathbb{Z}$.

c) An abelian group without elements of finite order B is called **slender**, if every homomorphisms $\psi : G \longrightarrow B$ maps all but a finite number of A_i to the identity of B. Prove that $\mathbb{Z}$ is slender.

d) Prove that there is no epimorphism of G onto H.