Homework 3

due on Friday, March 22

Read carefully the notes 4. and 5. at the bottom of the course web page. Read carefully sections 1.7, 2.2, 4.1-4.5, 5.4, 5.5 in Dummit & Foote. Solve problems 7, 9 to section 4.1, 7 to 4.2, 23, 24, 26, 27 to 4.3, 12 to 4.4, 16, 22, 51 to 4.5, 18 to 5.4, and 6, 8 to 5.5. Read Chapters 3,4,5 in Milne's book.

Solve the following problems.

Problem 1. Let $G = \sum_{i=-\infty}^{\infty} C$, where $C = \{0,1\}$ is the group of order 2. Thus G consists of all functions $f : \mathbb{Z} \longrightarrow C$ such that f(i) = 0 for all but finitely many i. Define an automorphism $t : G \longrightarrow G$ by (tf)(i) = f(i-1) (so it is a shift). Let $\phi : \mathbb{Z} \longrightarrow$ AutG be given by $\phi(1) = t$. Set $H = G \rtimes \mathbb{Z}$.

a) Prove that G is not finitely generated and that H is generated by (0,1) and (g,0) where g(0) = 1 and g(i) = 0 for all $i \neq 0$.

b) Show that [H, H] is the subgroup of G which consists of all f for which f(i) = 1 for an even number of i. Deduce that H, H is not finitely generated.

c) Conclude that the derived group of a free group on two generators is not finitely generated (this is true for any nonabelian free group).

Problem 2. Let G be a finite group and p a prime number such that p||G|. Consider the set S of all p-tuples $(a_1, ..., a_p)$ of elements from G such that $a_1a_2...a_p = e$, i.e.

$$S = \{(a_1, ..., a_p) : a_i \in G \text{ for all } i, \text{ and } a_1 ... a_p = e\}$$

Let C be a cyclic group of order p and f a generator for C. We define an action of C on S as follows: if $s \in C$ then $s = f^i$ for a unique $0 \le i < p$ and we set $s * (a_1, ..., a_p) = (a_{i+1}, a_{i+1}, ..., a_p, a_1, a_2, ..., a_i)$.

a) Check that this is indeed an action of C on S.

b) Show that the number of elements in S equals $|G|^{p-1}$.

c) Show that each fixed point for this action is of the form (g, ..., g) for some $g \in G$ such that $g^p = e$.

d) Conclude that G has a nontrivial element of order p (so we get a different proof of Cuchy's theorem).