Correction to Homework 4

Problem 22 to 6.1 in Dummit and Foote is incorrect as stated. Here is a counter-example.

Let $A = \mathbb{Q} \oplus \mathbb{Q}$ and let $H = SL_2(\mathbb{Q})$. Then H acts naturally on A and we can form the semi-direct product $G = A \rtimes H$. Note that $A \lhd G$.

Clam 1. $\Phi(A) = A$. Indeed, A has no maximal subgroups. To see this note that A is abelian so any maximal subgroup M of A is normal and A/M is an abelian simple group, i.e. a cyclic group of prime order p. On the other hand, A is divisible, so for any $a \in A$ there is $b \in A$ such that $pb = a$. It follows that $a + M = (pb) + M = p(b + M) = 0$. Thus $A/M = \{0\}$, a contradiction.

Claim 2. H is a maximal subgroup of G (where H is identified with the set of elements of the form $(0, L)$, $L \in H$) . In fact, if $H < K$, then there is $(a, L) \in K$ with $a \neq 0$, $a \in A$ and $L \in H$. Thus $(a, I) = (a, L)(0, L^{-1})$ is also in K. Since H acts transitively on non-zero elements from A, given any $b \in A$ there is $L_1 \in H$ such that $L_1(a) = b$. This means that in G we have $(b, I) = (0, L_1)(a, I)(0, L_1)^{-1} \in K$. This shows that A is contained in K. Thus both A and H are contained in K, i.e. $K = G$.

Claim 2 tells us that $\Phi(G)$ is contained in H. Thus $\Phi(G) \cap \Phi(A) = \{(0, I)\}$ is trivial. In particular, $\Phi(A)$ is not contained in $\Phi(G)$.

Exercise. What is $\Phi(G)$?

The conclusion of the problem is true if $\Phi(N)$ is finitely generated. Indeed, suppose that $\Phi(N)$ is generated by $a_1, \dots, a_k$. Note that each a_i is a non-generator for N. It follows that if $N =< a_1, \dots, a_k, S >$ for some subset S then $N =< S >$. In particular, if T is a subgroup of N such that $N = T\Phi(N)$ then $N =< T, a_1, \dots, a_k >$, so $T = N$.

Suppose H is a maximal subgroup of G which does not contain $\Phi(N)$. Since $\Phi(N)$ is characteristic in N, it is normal in G. Thus $H\Phi(N)$ is a subgroup of G properly containing H, so $G = H\Phi(N)$. It follows that $N = (H \cap N)\Phi(N)$, hence $N = H \cap N$ and $\Phi(N) \subseteq N \subseteq H$, a contradiction.