Problem 22 to 6.1 in Dummit and Foote is incorrect as stated. Here is a counter-example.

Let \( A = \mathbb{Q} \oplus \mathbb{Q} \) and let \( H = SL_2(\mathbb{Q}) \). Then \( H \) acts naturally on \( A \) and we can form the semi-direct product \( G = A \rtimes H \). Note that \( A \triangleleft G \).

**Claim 1.** \( \Phi(A) = A \). Indeed, \( A \) has no maximal subgroups. To see this note that \( A \) is abelian so any maximal subgroup \( M \) of \( A \) is normal and \( A/M \) is an abelian simple group, i.e. a cyclic group of prime order \( p \). On the other hand, \( A \) is divisible, so for any \( a \in A \) there is \( b \in A \) such that \( pb = a \). It follows that \( a + M = (pb) + M = p(b + M) = 0 \). Thus \( A/M = \{0\} \), a contradiction.

**Claim 2.** \( H \) is a maximal subgroup of \( G \) (where \( H \) is identified with the set of elements of the form \((0, L), L \in H\) ). In fact, if \( H < K \), then there is \((a, L) \in K \) with \( a \neq 0 \), \( a \in A \) and \( L \in H \). Thus \((a, I) = (a, L)(0, L^{-1})\) is also in \( K \). Since \( H \) acts transitively on non-zero elements from \( A \), given any \( b \in A \) there is \( L_1 \in H \) such that \( L_1(a) = b \). This means that in \( G \) we have \((b, I) = (0, L_1)(a, I)(0, L_1)^{-1} \in K \). This shows that \( A \) is contained in \( K \). Thus both \( A \) and \( H \) are contained in \( K \), i.e. \( K = G \).

Claim 2 tells us that \( \Phi(G) \) is contained in \( H \). Thus \( \Phi(G) \cap \Phi(A) = \{(0, I)\} \) is trivial. In particular, \( \Phi(A) \) is not contained in \( \Phi(G) \).

**Exercise.** What is \( \Phi(G) \)?

The conclusion of the problem is true if \( \Phi(N) \) is finitely generated. Indeed, suppose that \( \Phi(N) \) is generated by \( a_1, \ldots, a_k \). Note that each \( a_i \) is a non-generator for \( N \). It follows that if \( N = < a_1, \ldots, a_k, S > \) for some subset \( S \) then \( N = < S > \). In particular, if \( T \) is a subgroup of \( N \) such that \( N = T\Phi(N) \) then \( N = < T, a_1, \ldots, a_k > \), so \( T = N \).

Suppose \( H \) is a maximal subgroup of \( G \) which does not contain \( \Phi(N) \). Since \( \Phi(N) \) is characteristic in \( N \), it is normal in \( G \). Thus \( H\Phi(N) \) is a subgroup of \( G \) properly containing \( H \), so \( G = H\Phi(N) \). It follows that \( N = (H \cap N)\Phi(N) \), hence \( N = H \cap N \) and \( \Phi(N) \subseteq N \subseteq H \), a contradiction.