Homework 2

due on Monday, February 28

Read carefully Chapter 2 in Milne's book. Solve problems 2-3, 2-7, 2-9. Read carefully sections 5.1, 5.2, 6.3 in Dummit & Foote. Solve problem 36 to section 3.1, problem 10 to section 4.1, problems 3, 8 to section 5.4.

Solve the following problems.

Problem 1. Let H be a subgroup of $\mathbb{Z}^4$ generated by $(-1,-2,-3,-4)$, $(3,8,5,6)$, $(-1,0,-13,-16)$, $(-3,-4,-13,-6)$. Find a compatible bases of $\mathbb{Z}^4$ and H, then find the rank, the invariant factors, and the elementary divisors of the group $\mathbb{Z}^4/H$.

Problem 2. Let n be a positive integer. An element a in an abelian group A is called n-divisible if $a = nb$ for some $b \in A$. Let p be a prime. We say that a has infinite p-height if a is divisible by every power of p.

Consider the group $P = \prod_{i=1}^{\infty} < x_i >$, where x_i has order p^i. Let T be the torsion subgroup of P. Prove that no non-zero element of P has infinite height. Let $a = (a_i)$ be the element of P with $a_{2i} = x_{2i}^{p^i}$ and $a_{2i-1} = 0$ for $i = 1, 2, \ldots$. Prove that a is not in T and that $a + T$ has infinite p-height in P/T. Conclude that there is no subgroup B in P such that $P = T \oplus B$.

Problem 3. a) Let K, M, N be groups such that K is finite and $K \times M$ is isomorphic to $K \times N$. Prove that M and N are isomorphic.

Hint. Use induction on the order of K. Assume that $G = K_1H_1 = K_2H_2$, where K_1, H_1, K_2, H_2 are normal subgroups of G such that $K_1 \cap H_1 = \{e\} = K_2 \cap H_2$, K_1 and K_2 are both isomorphic to K, H_1 is isomorphic to M and H_2 is isomorphic to N. Consider $T_1 = K_1 \cap H_2$ and $T_2 = K_2 \cap H_1$. Prove that $K_1/T_1 \times K_2/T_2 \times H_1$ and $K_1/T_1 \times K_2/T_2 \times H_2$ are isomorphic.

b) Show that there are infinite groups K for which part a) is false.

Problem 4. For finite groups G, H define $h(G, H)$ to be the number of homomorphisms from G to H and $e(G, H)$ to be the number of injective homomorphisms from G to H.

a) Prove that $h(G, H) = \sum_N e(G/N, H)$, where the sum is over all normal subgroups N of G.

b) Suppose that H_1, H_2 are finite groups such that $h(G, H_1) = h(G, H_2)$ for every finite group G. Prove that H_1 and H_2 are isomorphic.

c) Use b) to give a different proof of a) in problem 3 when all groups are finite.

Problem 5. a) Let $m_1, \ldots, m_n$ be integers whose greatest common divisor is 1. Prove that $\mathbb{Z}^n$ has a basis whose first element is $(m_1, \ldots, m_n)$. Conclude that there is a matrix in $SL_n(\mathbb{Z})$ whose first row is $(m_1, \ldots, m_n)$.

b) Let $w_1, \ldots, w_n$ generate an abelian group A and let $M = (m_{i,j}$ be a matrix in $GL_n(\mathbb{Z})$. Show that the elements $u_i = \sum_j m_{i,j}w_j$ also generate A (compare to Lemma 1.53 in Miln's book).

Problem 6 Let $i \neq j$. An elementary matrix $E_{i,j}(k)$ is a square matrix which has all diagonal entries equal to 1, the i, j-entry equal to k, and all other entries equal to 0 (note that we do not specify the size of the matrix; it will follow from the context). Consider an $m \times n$ matrix $M \neq 0$ with integer entries.

a) Show that $E_{i,j}(s)E_{i,j}(t) = E_{i,j}(s+t)$.

b) Let $i \neq j$. Show that there is a product of elementary matrices U such that UM is obtained from M by replacing the i-th row of M with the j-th row of M and the j-th row of M with the negative of the i-th row of M.

c) Let $i \neq j$. Show that there is a product of elementary matrices D such that DM is obtained from M by multiplying both the i-th and j-th rows of M by -1.

d) Prove that there are products of elementary matrices A, B and $s \leq \min(m,n)$ such that AMB is a matrix $(a_{i,j})$ such that the only non-zero entries are $a_{1,1}, a_{2,2}, \ldots, a_{s,s}$ and $a_{1,1}|a_{2,2}|\ldots|a_{s,s}$ and $a_{2,2}, \ldots, a_{s,s}$ are positive.

e) Use d) to show that every matrix in $SL_n(\mathbb{Z})$ is a product of elementary matrices. Conclude that $SL_n(\mathbb{Z})$ is finitely generated.

Remark. a), b), c) are true for matrices over any ring. For d) apply the method we used in class to prove the result about compatible bases for a free abelian group of finite rank and its subgroup. First part of e) is also true for matrices over any field

Problem 7 Observe that if A, C are abelian groups then the set $\mathrm{Hom}(A,C)$ of all group homomorphisms from A to C is also an abelian group with addition defined by $(f+g)(a) = f(a)+g(a))$ (verify this).

a) Let A, B, C be abelian groups. Show that the group $\mathrm{Hom}(A \times B, C)$ is naturally isomorphic to $\mathrm{Hom}(A,C) \times \mathrm{Hom}(B,C)$.

b) Suppose that A is a cyclic group of order which divides n and C is a cyclic group of order n. Prove that the groups $\mathrm{Hom}(A,C)$ and A are isomorphic.

c) Use the structure theorem for finite abelian groups to show that if A is abelian of exponent dividing n and C is cyclic of order n then the groups $\mathrm{Hom}(A,C)$ and A are isomorphic.

d) Let A, B, C be abelian groups. A map $F : A \times B \longrightarrow C$ is called **bilinear** if for any $a \in A$ the map $F(a,-) : b \mapsto F(a,b)$ is a homomorphism from B to C and for any $b \in B$ the map $F(-,b) : a \mapsto F(a,b)$ is a homomorphism from A to C. Show that if F is bilinear then the assignment $a \mapsto F(a,-)$ defines a homomorphism from A to $\mathrm{Hom}(B,C)$ and the assignment $b \mapsto F(-,b)$ defines a homomorphism from B to $\mathrm{Hom}(A,C)$. We say that F is **non-degenerate** if these two homomorphisms are injective.

d) Let C be a cyclic group of order n. Suppose that A, B are finite abelian groups and $F : A \times B \longrightarrow C$ is a non-degenrate bilinear map (we ofetn say that F is a non-degenerate pairing in this situation). Prove that the groups A and $\mathrm{Hom}(B,C)$ are isomorphic. Conclude that A and B are isomorphic and of exponent which divides n.

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Challenge. Let G be the product of countable many copies of $\mathbb{Z}$, i.e. $G = \Pi_{i=1}^{\infty} A_i$, where $A_i = \mathbb{Z}$ for all i. Let H be the direct sum of these groups.

a) Prove that if $\phi : G \longrightarrow \mathbb{Z}$ is a homomorphism such that $H < \ker\phi$, then $\ker\phi = G$.

b) Prove that G is not isomorphic to a direct sum of the form $\sum_{i\in I} \mathbb{Z}$.

c) An abelian group without elements of finite order B is called **slender**, if every homomorphisms $\psi : G \longrightarrow B$ maps all but a finite number of A_i to the identity of B. Prove that $\mathbb{Z}$ is slender.

d) Prove that there is no epimorphism of G onto H.