
Solve the following problems.

**Problem 1.** Let $H$ be a subgroup of $\mathbb{Z}^4$ generated by $(-1, -2, -3, -4), (3, 8, 5, 6), (-1, 0, -13, -16), (-3, -4, -13, -6)$. Find a compatible bases of $\mathbb{Z}^4$ and $H$, then find the rank, the invariant factors, and the elementary divisors of the group $\mathbb{Z}^4/H$.

**Problem 2.** Let $n$ be a positive integer. An element $a$ in an abelian group $A$ is called $n$-divisible if $a = nb$ for some $b \in A$. Let $p$ be a prime. We say that $a$ has infinite $p$-height if $a$ is divisible by every power of $p$.

Consider the group $P = \prod_{i=1}^{\infty} x_i$, where $x_i$ has order $p^i$. Let $T$ be the torsion subgroup of $P$. Prove that no non-zero element of $P$ has infinite height. Let $a = (a_i)$ be the element of $P$ with $a_{2i} = x_{2i}^{p^i}$ and $a_{2i-1} = 0$ for $i = 1, 2, \ldots$. Prove that $a$ is not in $T$ and that $a + T$ has infinite $p$-height in $P/T$. Conclude that there is no subgroup $B$ in $P$ such that $P = T \oplus B$.

**Problem 3.** a) Let $K, M, N$ be groups such that $K$ is finite and $K \times M$ is isomorphic to $K \times N$. Prove that $M$ and $N$ are isomorphic.

**Hint.** Use induction on the order of $K$. Assume that $G = K_1H_1 = K_2H_2$, where $K_1, H_1, K_2, H_2$ are normal subgroups of $G$ such that $K_1 \cap H_1 = \{e\} = K_2 \cap H_2$, $K_1$ and $K_2$ are both isomorphic to $K$, $H_1$ is isomorphic to $M$ and $H_2$ is isomorphic to $N$. Consider $T_1 = K_1 \cap H_2$ and $T_2 = K_2 \cap H_1$. Prove that $K_1/T_1 \times K_2/T_2 \times H_1$ and $K_1/T_1 \times K_2/T_2 \times H_2$ are isomorphic.

b) Show that there are infinite groups $K$ for which part a) is false.

**Problem 4.** For finite groups $G, H$ define $h(G, H)$ to be the number of homomorphisms from $G$ to $H$ and $e(G, H)$ to be the number of injective homomorphisms from $G$ to $H$.

a) Prove that $h(G, H) = \sum_N e(G/N, H)$, where the sum is over all normal subgroups $N$ of $G$.

b) Suppose that $H_1, H_2$ are finite groups such that $h(G, H_1) = h(G, H_2)$ for every finite group $G$. Prove that $H_1$ and $H_2$ are isomorphic.

c) Use b) to give a different proof of a) in problem 3 when all groups are finite.

**Problem 5.** a) Let $m_1, \ldots, m_n$ be integers whose greatest common divisor is 1. Prove that $\mathbb{Z}^n$ has a basis whose first element is $(m_1, \ldots, m_n)$. Conclude that there is a matrix in $SL_n(\mathbb{Z})$ whose first row is $(m_1, \ldots, m_n)$.

b) Let $w_1, \ldots, w_n$ generate an abelian group $A$ and let $M = (m_{i,j}$ be a matrix in $GL_n(\mathbb{Z})$. Show that the elements $u_i = \sum_j m_{i,j}w_j$ also generate $A$ (compare to Lemma 1.53 in Milne’s book).

**Problem 6** Let $i \neq j$. An elementary matrix $E_{i,j}(k)$ is a square matrix which has all diagonal entries equal to 1, the $i, j$-entry equal to $k$, and all other entries equal to 0 (note that we do
not specify the size of the matrix; it will follow from the context). Consider an $m \times n$ matrix $M \neq 0$ with integer entries.

a) Show that $E_{i,j}(s)E_{i,j}(t) = E_{i,j}(s + t)$.

b) Let $i \neq j$. Show that there is a product of elementary matrices $U$ such that $UM$ is obtained from $M$ by replacing the $i$-th row of $M$ with the $j$-th row of $M$ and the $j$-th row of $M$ with the negative of the $i$-th row of $M$.

c) Let $i \neq j$. Show that there is a product of elementary matrices $D$ such that $DM$ is obtained from $M$ by multiplying both the $i$-th and $j$-th rows of $M$ by $-1$.

d) Prove that there are products of elementary matrices $A$, $B$ and $s \leq \min(m,n)$ such that $AMB$ is a matrix $\begin{pmatrix} a_{i,j} \end{pmatrix}$ such that the only non-zero entries are $a_{1,1}, a_{2,2}, \ldots, a_{s,s}$ and $a_{1,1}|a_{2,2}|\ldots|a_{s,s}$ and $a_{2,2}, \ldots, a_{s,s}$ are positive.

e) Use d) to show that every matrix in $SL_n(\mathbb{Z})$ is a product of elementary matrices. Conclude that $SL_n(\mathbb{Z})$ is finitely generated.

**Remark.** a), b), c) are true for matrices over any ring. For d) apply the method we used in class to prove the result about compatible bases for a free abelian group of finite rank and its subgroup. First part of e) is also true for matrices over any field.

**Problem 7** Observe that if $A, C$ are abelian groups then the set $\text{Hom}(A, C)$ of all group homomorphisms from $A$ to $C$ is also an abelian group with addition defined by $(f + g)(a) = f(a) + g(a)$ (verify this).

a) Let $A, B, C$ be abelian groups. Show that the group $\text{Hom}(A \times B, C)$ is naturally isomorphic to $\text{Hom}(A, C) \times \text{Hom}(B, C)$.

b) Suppose that $A$ is a cyclic group of order which divides $n$ and $C$ is a cyclic group of order $n$. Prove that the groups $\text{Hom}(A, C)$ and $A$ are isomorphic.

c) Use the structure theorem for finite abelian groups to show that if $A$ is abelian of exponent dividing $n$ and $C$ is cyclic of order $n$ then the groups $\text{Hom}(A, C)$ and $A$ are isomorphic.

d) Let $A, B, C$ be abelian groups. A map $F : A \times B \rightarrow C$ is called **bilinear** if for any $a \in A$ the map $F(a, -) : b \mapsto F(a, b)$ is a homomorphism from $B$ to $C$ and for any $b \in B$ the map $F(-, b) : a \mapsto F(a, b)$ is a homomorphism from $A$ to $C$. Show that if $F$ is bilinear then the assignment $a \mapsto F(a, -)$ defines a homomorphism from $A$ to $\text{Hom}(B, C)$ and the assignment $b \mapsto F(-, b)$ defines a homomorphism from $B$ to $\text{Hom}(A, C)$. We say that $F$ is **non-degenerate** if these two homomorphisms are injective.

d) Let $C$ be a cyclic group of order $n$. Suppose that $A, B$ are finite abelian groups and $F : A \times B \rightarrow C$ is a non-degenerate bilinear map (we often say that $F$ is a non-degenerate pairing in this situation). Prove that the groups $A$ and $\text{Hom}(B, C)$ are isomorphic. Conclude that $A$ and $B$ are isomorphic and of exponent which divides $n$.

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**Challenge.** Let $G$ be the product of countable many copies of $\mathbb{Z}$, i.e. $G = \prod_{i=1}^{\infty} A_i$, where $A_i = \mathbb{Z}$ for all $i$. Let $H$ be the direct sum of these groups.

a) Prove that if $\phi : G \rightarrow \mathbb{Z}$ is a homomorphism such that $H < \ker \phi$, then $\ker \phi = G$. 

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b) Prove that $G$ is not isomorphic to a direct sum of the form $\sum_{i \in I} \mathbb{Z}$.

c) An abelian group without elements of finite order $B$ is called slender, if every homomorphisms $\psi : G \to B$ maps all but a finite number of $A_i$ to the identity of $B$. Prove that $\mathbb{Z}$ is slender.

d) Prove that there is no epimorphism of $G$ onto $H$. 