Homework 3

due on Monday, March 14

Read carefully sections 1.7, 2.2, 4.1-4.5, 5.4, 5.5 in Dummit & Foote. Solve problems 7, 9 to section 4.1, 7 to 4.2, 23, 24, 26, 27 to 4.3, 12 to 4.4, 18 to 5.4, and 6, 8 to 5.5. Read Chapters 3,4,5 in Milne's book.

Solve the following problems.

Problem 1. Let z be a complex number such that $|z| \geq 2$. In the group $GL_2(\mathbb{C})$ of all 2×2 invertible matrices with complex entries consider the matrices $a = E_{1,2}(z)$ and $b = E_{2,1}(z)$ (see problem 6 of homework 2). Prove that the subgroup generated by $\{a, b\}$ is a free group on the set $\{a, b\}$.

Problem 2. We consider the group $G = SL_2(\mathbb{Z})$. From Problem 6 in homework 2, G is generated by the two matrices $a = E_{1,2}(1)$ and $b = E_{2,1}(1)$. We proved in class that the subgroup H generated by $\{a^2, b^2\}$ is a free group on the set $\{a^2, b^2\}$.

a) Show that H is not normal in G. Hint: Consider $a^{-1}(a^2b^{-2})a$.

b) Prove that $K = H \cdot \{I, -I\}$ is a normal subgroup of G and G/K is either abelian or dihedral.

c) Let $x = ab^{-1}$ and $y = ab^{-1}a$. Prove that $\{x, y\}$ generates G. What are the orders of x and y? Conclude that G/K is isomorphic to D_3 - the dihedral group of order 6. What is $[G : H]$?

d) Consider the natural homomorphism $\phi : SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/2\mathbb{Z})$. Prove that ϕ surjective and K is the kernel of ϕ.

Problem 3. Let $G = \bigoplus_{i=-\infty}^{\infty} C_i$, where $C_i = \{0, 1\}$ is the group of order 2 for every $i \in \mathbb{Z}$. Thus G consists of all functions $f : \mathbb{Z} \longrightarrow \{0, 1\}$ such that $f(i) = 0$ for all but finitely many i. Define an automorphism $t : G \longrightarrow G$ by $(tf)(i) = f(i-1)$ (so it is a shift). Let $\phi : \mathbb{Z} \longrightarrow \mathrm{Aut}G$ be given by $\phi(1) = t$. Define H to be the semidirect product $H = G \rtimes \mathbb{Z}$.

a) Prove that G is not finitely generated and that H is generated by $(0, 1)$ and $(g, 0)$ where $g(0) = 1$ and $g(i) = 0$ for all $i \neq 0$.

b) Show that $[H, H]$ is the subgroup of G which consists of all f for which $f(i) = 1$ for an even number of i. Deduce that $[H, H]$ is not finitely generated.

c) Conclude that the derived group of a free group on two generators is not finitely generated (this is true for any nonabelian free group).

Problem 4. Let G be a finite group and p a prime number such that $p||G|$. Consider the set S of all $p-$tuples $(a_1, ..., a_p)$ of elements from G such that $a_1a_2...a_p = e$, i.e.

$$S = \{(a_1, ..., a_p) : a_i \in G \text{ for all } i, \text{ and } a_1...a_p = e\}$$

Let C be a cyclic group of order p and f a generator for C. We define an action of C on S as follows: if $s \in C$ then $s = f^i$ for a unique $0 \leq i < p$ and we set $s * (a_1, ..., a_p) = (a_{i+1}, a_{i+1}, ...a_p, a_1, a_2, ..., a_i)$.

a) Check that this is indeed an action of C on S.

b) Show that the number of elements in S equals $|G|^{p-1}$.

c) Show that each fixed point for this action is of the form $(g, ..., g)$ for some $g \in G$ such that $g^p = e$.

d) Conclude that G has a nontrivial element of order p (so we get a different proof of Cauchy's theorem).