Homework 4

due on Monday, April 4

Solve problem 18 to section 4.4, 46 to 4.5, 10, 21 to 5.5, and 21, 22, 24, 26 to 6.1.

Solve the following problems.

Problem 1. Let p be the smallest prime divisor of the order of a finite group G. Prove that if $H < G$ and $[G : H] = p$ then H is normal in G.

Problem 2. Let P be a Sylow p-subgroup of G and let $H < G$. Consider the natural action of H on the left cosets of P in G ($h(aP) = (ha)P$). Show that the stabilizer of some left coset is a Sylow p-subgroup of H. (This result can be used to give a different proof of existence of Sylow p-subgroups).

Problem 3. Prove that there are no simple groups of order 2025.

Problem 4. a) Prove that S_n is generated by $(1, 2)$ and $(1, 2, \ldots, n)$.

b) Show that S_n is generated by $(1, 2), (2, 3), \ldots, (n - 1, n)$.

c) Show that if p is prime then S_p is generated by any set $\{a, b\}$, where a is a transposition and b has order p.

d) Let $d|n$ and $1 < d < n$. Show that S_n is NOT generated by $(1, 2, \ldots, n)$ and $(1, d + 1)$

Problem 5. a) Prove that if P is a Sylow p-subgroup of G and N is normal in G then $N \cap P$ is a Sylow p-subgroup of N.

b) If $f : G \longrightarrow H$ is a surjective homomorphism and P is a Sylow p-subgroup of G then $f(P)$ is a Sylow p-subgroup of H.

Problem 6. Let P be a Sylow p-subgroup of G and M a subgroup of G which contains the normalizer $N_G(P)$. Prove that $N_G(M) = M$. Consider the conjugation action of P on the set of all subgroups conjugate in G to M. Show that M is the only fixed point. Conclude that $[G : M] \equiv 1 (\mod p)$.