Recall from previous lectures:

**Theorem 1:** Let $G$ be a nilpotent group of class $k$ and $a \in G$. Then $[G, G] \cdot \langle a \rangle$ is a normal subgroup of $G$ of class $\leq k-1$.

We used this result to prove the following (by induction on class):

**Theorem 2:** Let $G$ be nilpotent. The set $T(G)$ of all torsion elements (i.e., of finite order) is a characteristic subgroup of $G$.

**Exercise:** Let $m \in \mathbb{N}$. Use the same technique to prove that the set of all $g \in G$ such that $g^m = e$ for some $k$ is a subgroup.

Conclude that:

**Corollary 1:** $G$ nilpotent then $T(G) = \bigoplus_p T_p(G)$, where $T_p(G)$ is the set of all elements of $p$-power order.

Last time we also proved that:

**Theorem 3:** $G$ nilpotent, $d_4 \neq H \leq G \Rightarrow H \cap G \neq d_4$.

**Corollary 2:** $G$ nilpotent, then

(a) $G$ has element of order $p$ iff $G_1(G)$ has element of order $p$ ($p$ a prime)

(b) $G$ is torsion free $\iff G_1(G)$ is torsion free.

**Proof:** By Theorem 3, $T_p(G) \neq d_4 \iff T_p(G)/G_1(G) \neq d_4$. This proves (a).

Also, $T(G) \neq d_4 \iff T(G)/G_1(G) \neq d_4$, which proves (b).

**Theorem 4:** Any group. Then

(i) $G_1(G)$ has no elements of order $p$ $\implies G_1(G)$ has no elements of order $p$ (here $p$ is a prime).

(ii) $G_1(G)$ has exponent $m$ $\implies G_1(G)$ has exponent dividing $m$. 

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Proof: Recall that \( y_2(5) / y_1(5) = y_1(5) / y_2(5) \). In particular, if \( g \in G \) and \( a \in y_2(5) \), then \( [a,b] \in y_1(5) \). It follows that for \( g \in y_2(5) \) we have \( [a,b]^n = [a,g]^n [g,b]^n = [a,g][b,g] \) (as \( [a,g] \in y_1(5) \)). It follows that \( [a,y]^n = [a,g]^n \) for any \( a \in y_2(5), g \in G\). 

Suppose \( y_1(5) \) has no elements of order \( p \) and consider \( a \in y_2(5) \) such that \( a^p \in y_2(5) \) (i.e., \( a^p \in y_1(5) \)). Then for any \( g \in G \) we have \( e = [a,y]^n = [a,g]^n \). Since \( [a,g] \in y_1(5) \), we conclude that \( [a,y]^n = e \). Since \( g \) was arbitrary, \( a \in y_2(5) \). This proves (1).

Suppose now \( n^m = e \) for all \( a \in y_2(5) \). Then for \( a \in y_2(5), g \in G \) we have \( [a,y]^n = [a,g]^n = e \), so \( a^m \in y_1(5) \). This proves (2).

Corollary 3. Let \( G \) be nilpotent. Following conditions are equivalent:

1. \( G \) is torsion-free
2. \( y_1(5) \) is torsion-free
3. \( y_k(5) / y_{k+1}(5) \) is torsion-free for all \( k \).

Proof: That (1) \( \Rightarrow \) (2) follows from Corollary 2. Clearly (2) \( \Rightarrow \) (3) for \( k = 1 \). Now recall that \( y_k(5) / y_{k+1}(5) = \frac{y_1(5)}{y_k(5) / y_{k+1}(5)} \). Thus (3) \( \Rightarrow \) (2) follows from (2) of Theorem 4 and obvious induction.

(Note that a group is torsion-free if and only if every prime \( p \) does not have elements of order \( p \).)

Corollary 4: Let \( G \) be nilpotent of rank \( R \). If \( y_1(5) \) has exponent \( m \), then \( G \) has exponent dividing \( m^4 \).
Proof: As in the proof of Corollary 3, we see that \( g(6) \) has exponent \( m \) implies that \( g(6)^m \) has exponent dividing \( m \) for all \( k \). This means that \( g \in g(6) \) then \( g^m \in g(6) \). Starting with any \( g \in G = g(6) \) we see that
\[
g^m \in g(6) = \ker f.
\]

Another application of Theorem 1 is the following

**Theorem 5:** Let \( G \) be a torsion-free nilpotent group. If \( g, b \in G \) and \( a^m = b^m \) then \( a = b \).

**Proof:** We use induction on the class of \( G \). If \( G \) is abelian and torsion-free the result is clear: \( a^m = b^m \Rightarrow (a^{-1}b)^m = de \Rightarrow a^{-1}b = e \Rightarrow a = b \).

Suppose the result holds for groups of class \( < k \) and let \( G \) be of class \( k \), \( g, b \in G \) and \( a^m = b^m \). The group \([g, b] \) has class \( < k \) and \( [b, b^{-1}] = (b b^{-1})^m \). Also,
\[
(b b^{-1})^m = b^m b^{-1} = b^m b^{-1} = b^m = a^m.
\]
By inductive assumption, \( b b^{-1} = a \), i.e. \( a \) and \( b \) commute. Thus \( (a^{-1}b)^m = a^{-1}b = \ker f \) and therefore \( a^{-1}b = e \), i.e. \( a = b \).

We end by discussing some basic properties of finitely generated nilpotent groups. Recall that we proved that if \( G \) is finitely generated then \( G(6)^m \) is finitely generated for all \( k \). It follows that \( G = G(6)^2 \neq G(6)^2 \). - 2 \( G(6)^2 \) = \( \ker f \) is a central series with finitely generated abelian successive quotients. Since a refinement of a central series is central, we see

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that $G$ has a central series with cyclic successive quotients. Since central series is normal we see that

Theorem 6: A finitely generated nilpotent group is supersolvable (hence also polycyclic). In particular, every subgroup of a finitely generated nilpotent group is finitely generated.

Theorem 7: A finitely generated torsion nilpotent group is finite.

**Proof.** Induction on class. If $G$ is abelian, finitely generated and torsion, then $G$ is finite. Suppose works for groups of class $< k$ and consider a finitely generated torsion nilpotent group of class $k$. Then $G/G_k$ has class $< k-1$ and is finitely generated and torsion, so $G/G_k$ is finite. Also $G/G_k$ is finitely generated, abelian and torsion, hence finite. Since both $G/G_k$ and $G/G_k$ are finite, so is $G$.

Corollary 5: If $G$ is finitely generated nilpotent then $T(G)$ is finite. Note that $G/T(G)$ is torsion-free. It is **false** in general that $T(G)$ is a direct factor of $G$. However, the following is true:

Theorem 8: A finitely generated nilpotent group has a torsion-free subgroup of finite index.

**Proof.** Induction on class of $G$. If $G$ is abelian this follows from classification of finitely generated abelian groups. Suppose the result holds for groups of class $< k$ and let $G$ be of class $k$. 

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Then $\Gamma_2(G)$, being of class $< k$, has a finite index subgroup $M$ which is torsion-free. The intersection $\cap M$ of all subgroups of $\Gamma_2(G)$ of index $[G: H]$ is then a characteristic, torsion-free subgroup of $\Gamma_2(G)$ of finite index. Consider $G/H$. The commutator of $G/H$ is $\Gamma_2(G/H)$, which is finite. It follows that the center of $G/H$ has finite index and therefore $G/H$ has a subgroup $H'$ of finite index which is torsion-free. Then $H'$ is of finite index in $G$ and torsion-free. $\Box$

We end by listing without proof some deeper results.

**Theorem A:** Every finitely generated torsion-free nilpotent group can be embedded into $\text{UT}_n(\mathbb{Z})$ for some $n$.

**Theorem B:**
1. If $G$ is supersolvable then $[G, G]$ is nilpotent.
2. Any polycyclic group $G$ has a subgroup $H$ of finite index such that $H$ is torsion-free and $[H, H]$ is nilpotent.

**Theorem C (Malcev):** A soluble subgroup of $\text{GL}_n(\mathbb{Z})$ is polycyclic.

**Theorem D (Auslander–Swan):** Let $G$ be a polycyclic group with a torsion-free normal nilpotent subgroup $N$ such $G/N$ is abelian and torsion-free. Then there is injective homomorphism $\phi: G \to \text{GL}_n(\mathbb{Z})$ of $\text{GL}_n(\mathbb{Z})$ (for some $n$).

In particular, every polycyclic group can be embedded into $\text{GL}_n(\mathbb{Z})$ for some $n$. $\Phi$

THE END