Recall from previous lectures:

<u>Theorem 1</u>: Let G be a nilpotent group of class k and $a \in G$. Then $[G,G]\cdot\langle a\rangle$ is a normal subgroup of G of class $\leq k-1$.

We used this result to prove the following (by induction on class)

<u>Theorem 2</u>: Let G be nilpotent. The set $T(G)$ of all torsion elements (i.e. of finite order) is a characteristic subgroup of G.

<u>Exercise</u>: Let $m \in \mathbb{N}$. Use the same technique to prove that the set of all $g \in G$ such that $g^{m^k} = e$ for some k is a subgroup.
Conclude that:

<u>Corollary 1</u>: G nilpotent then $T(G) = \bigoplus_{p \text{ prime}} T_p(G)$, where $T_p(G)$ is the set of all elements of p-power order.

Last time we also proved that

<u>Theorem 3</u>: G nilpotent, $\{e\} \neq H \trianglelefteq G \Rightarrow H \cap \zeta_1(G) \neq \{e\}$.

<u>Corollary 2</u>: G nilpotent, then
(a) G has element of order p iff $\zeta_1(G)$ has element of order p (p a prime)
(b) G is torsion-free $\Leftrightarrow \zeta_1(G)$ is torsion free.

<u>Pf</u>: By theorem 3, $T_p(G) \neq \{e\} \Leftrightarrow T_p(G) \cap \zeta_1(G) \neq \{e\}$. This proves a)
Also, $T(G) \neq \{e\} \Leftrightarrow T(G) \cap \zeta_1(G) \neq \{e\}$, which proves b).

<u>Theorem 4</u>: G any group. Then
(1) $\zeta_1(G)$ has no elements of order $p \Rightarrow \zeta_2(G)/\zeta_1(G)$ has no elements of order p (here p is a prime)

(2) $\zeta_1(G)$ has exponent $m \Rightarrow \zeta_2(G)/\zeta_1(G)$ has exponent dividing m.

Proof: Recall that $Z_2(G)/Z_1(G) = Z_1\left(G/Z_1(G)\right)$. In particular, if $g \in G$ and $a \in Z_2(G)$ then $[a,g] \in Z_1(G)$. It follows that for $a,b \in Z_2(G)$ we have $[ab,g] = [a,g]^b[b,g] = [a,g][b,g]$ (as $[a,g] \in Z_1(G)$)
It follows that $[a^n,g] = [a,g]^n$ for any $a \in Z_2(G)$, $g \in G$, $n \in \mathbb{Z}$.

Suppose $Z_1(G)$ has no elements of order p and consider $a \in Z_2(G)$ such that $a^p \in Z_1(G)$ (i.e. $(aZ_1(G))^p = e$ in $Z_2(G)/Z_1(G)$). Then for any $g \in G$ we have $e = [a^p,g] = [a,g]^p$. Since $[a,g] \in Z_1(G)$, we conclude that $[a,g] = e$. Since g was arbitrary, $a \in Z_1(G)$. This proves (1).

Suppose now $u^m = e$ for all $u \in Z_1(G)$. Then for $a \in Z_2(G)$, $g \in G$ we have $[a^m,g] = [a,g]^m = e$, so $a^m \in Z_1(G)$. This proves (2).

Corollary 3. Let G be nilpotent. Following conditions are equivalent:

(1) G is torsion-free

(2) $Z_1(G)$ is torsion-free

(3) $Z_k(G)/Z_{k-1}(G)$ is torsion-free for all k.

Pf: That (1)$\iff$(2) follow from Corollary 2 b). Clearly (2) is (3) for $k=1$. Thus (3)$\Rightarrow$(2). Now recall that
$$Z_k(G)/Z_{k-1}(G) = Z_2\left(G/Z_{k-2}(G)\right) \Big/ Z_1\left(G/Z_{k-2}(G)\right)$$
Thus (3)$\Rightarrow$(2) follows from (1) of Theorem 4 and obvious induction (note that a group is torsion-free iff for every prime p it does not have elements of order p).

Corollary 4: Let G be nilpotent of rank k. If $Z_1(G)$ has exponent m then G has exponent dividing m^k.

Proof: As in the proof of Corollary 3, we see that $\gamma_1(G)$ has exponent m implies that $\gamma_t(G)/\gamma_{t+1}(G)$ has exponent dividing m for all t. This means that if $g \in \gamma_t(G)$ then $g^m \in \gamma_{t+1}(G)$. Starting with any $g \in G = \gamma_k(G)$ we see that

$$g^{m^k} \in \gamma_0(G) = \{e\}. \boxtimes$$

Another application of Theorem 1 is the following

Theorem 5: Let G be a torsion-free nilpotent group. If $a, b \in G$ and $a^n = b^n$ then $a = b$.

Pf. We use induction on the class of G. If G is abelian and torsion-free the result is clear: $a^n = b^n \Rightarrow (a^{-1}b)^n = \{e\} \Rightarrow a^{-1}b = e \Rightarrow a = b$.

Suppose the result holds for groups of class $< k$ and let G be of class k, $a, b \in G$ and $a^n = b^n$. The group $[G,G]\cdot\langle a\rangle$ has class $< k$ and $bab^{-1} = (bab^{-1}a^{-1})a \in [G,G]\cdot\langle a\rangle$. Also,

$$(bab^{-1})^n = ba^nb^{-1} = bb^nb^{-1} = b^n = a^n.$$ By inductive assumption, $bab^{-1} = a$, i.e. a and b commute. Thus $(a^{-1}b)^n = a^{-n}b^n = \{e\}$ and therefore $a^{-1}b = e$, i.e. $a = b$. $\boxtimes$

We end by discussing some basic properties of finitely generated nilpotent groups. Recall that we proved that if G is finitely generated then $\gamma_k(G)/\gamma_{k+1}(G)$ is finitely generated for all k.

It follows that $G = \gamma_1(G) \geq \gamma_2(G) \geq \dots \geq \gamma_{k+1}(G) = \{e\}$ is a central series with finitely generated abelian successive quotients. Since a refinement of a central series is central, we see that G has a central series with cyclic successive quotients. Since central series is normal we see that

<u>Theorem 6</u>: A finitely generated nilpotent group is supersolvable (hence also polycyclic). In particular, every subgroup of a finitely generated nilpotent group is finitely generated.

<u>Theorem 7</u>: A finitely generated torsion nilpotent group is finite.

<u>Pf.</u> Induction on class. If G is abelian, finitely generated and torsion then G is finite. Suppose works for groups of class $< k$ and consider a finitely generated torsion nilpotent group of class k. Then $\gamma_2(G) = [G,G]$ has class $\leq k-1$ and is finitely generated and torsion, so $\gamma_2(G)$ is finite. Also $G/\gamma_2(G)$ is finitely generated, abelian and torsion, hence finite. Since both $\gamma_2(G)$ and $G/\gamma_2(G)$ are finite, so is G.

<u>Corollary 5</u>: If G is finitely generated nilpotent then $T(G)$ is finite.

Note that $G/T(G)$ is torsion-free. It is <u>FALSE</u> in general that $T(G)$ is a direct factor of G. However, the following is true:

<u>Theorem 8</u>: A finitely generated nilpotent group has a torsion-free subgroup of finite index.

<u>Pf.</u> Induction on class of G. If G is abelian this follows from classification of finitely generated abelian groups. Suppose the result holds for groups of class $< k$ and let G be of class k.

Then $\gamma_2(G)$ being of class $<k$, has a finite index subgroup M which is torsion free. The intersection N of all subgroups of $\gamma_2(G)$ of index $[G:M]$ is then a characteristic, torsion-free subgroup of $\gamma_2(G)$ of finite index. Consider G/N. The commutator of G/N is $\gamma_2(G)/N$, which is finite. It follows that the center of G/N has finite index an therefore G/N has a subgroup H/N of finite index which is torsion-free. Then H is of finite index in G and torsion-free. $\boxtimes$

We end by listing without proof some deeper results.

<u>Theorem A</u>: Every finitely generated torsion-free nilpotent group can be embedded into $UT_n(\mathbb{Z})$ for some n.

<u>Theorem B</u>: (1) If G is supersolvable then $[G,G]$ is nilpotent.
(2) Any polycyclic group G has a subgroup H of finite index such that H is torsion-free and $[H,H]$ is nilpotent.

<u>Theorem C</u> (Malcev): A solvable subgroup of $GL_n(\mathbb{Z})$ is polycyclic.

<u>Theorem D</u> (Auslander-Swan): Let G be a polycyclic group with a torsion-free normal nilpotent subgroup N s.t. G/N is abelian and torsion-free. Then there is injective homomorphism $\varphi: G \longrightarrow GL_n(\mathbb{Z})$ s.t. $\varphi(N) \subseteq UT_n(\mathbb{Z})$ (for some n).

In particular, every polycyclic group can be embedded into $GL_n(\mathbb{Z})$ for some n.

<u>THE END</u>