Homework 1

due on Wednesday, February 7

Read carefully Chapter 1 of Miln's book and sections 1.1-1.6, 2.1, 2.3, 2.4, 2.5, 3.1, 3.2, 3.3 of Dummit and Foote.

Problem 1. Let G be a group. Recall that (m,n) is the greatest common divisor of m and n. Prove that:

a) If $a \in G$ has finite order n then, for any integer k, the order of a^k is $n/(n,k)$.

b) If a has order m, b has order n, and $ab = ba$ then the order of ab divides $mn/(m,n)$ and is divisible by $mn/(m,n)^2$.

c) If G has an element a of order m and an element b of order n such that $ab = ba$ then G has an element of order $[m,n]$ ($[m,n]$ is the least common multiple of m and n).

d) If G is a finite abelian group and N is the smallest positive integer such that $g^N = e$ for all $g \in G$, then G has an element of order N.

Remark. In general, the N defined in d) makes sense for any group (it can be infinite) and it is called **the exponent of** G.

e) If $f : G \longrightarrow H$ is a homomorphism and $a \in G$ has finite order n, then $f(a)$ has also finite order k which divides n. Also, a^m is in the kernel of f iff k divides m.

f) Let G be a cyclic group of order n and H a cyclic group of order m (we allow the orders to be infinite). Show that the set of all homomorphism from G to H is a group with operation $+$ defined by $(f+g)(a) = f(a)g(a)$ (this is true for arbitrary G and abelian H). Describe this group for each pair m, n.

g) Study Theorem 1.64 and its proof in Miln's book.

Problem2. Let G be a group and H its subgroup.

a) Show that if a_iH, $i \in I$ are all the left cosets of H in G then Ha_i^{-1}, $i \in I$ are all the right cosets of H in G (each listed once). Conclude that the number of left cosets of H is finite iff the number of right cosets is finite and these numbers coincide. The number of left (right) cosets of H in G is called the **index** of H in G and it is usually denoted by $[G : H]$.

b) Prove that if $K < H < G$ then $[G : K] = [G : H][H : K]$.

c) Show that for any subgroup K of G we have $[K : H \cap K] \leq [G : H]$.

d) Prove that if H, K are subgroups of G of finite index then so is $H \cap K$ and $[G : H \cap K] \leq [G : H][G : K]$.

e) Prove that if H is of finite index then G is finitely generated iff H is finitely generated.

f) Prove that if H is of finite index then there is a normal subgroup of G of finite index contained in H (show that the number of conjugates of H is finite and take their intersection).

g) Show that if G is finitely generated then it has only finitely many subgroups of a given finite index n (use the fact that the action of G on cosets of a subgroup K of index n defines a homomorphism of G into S_n whose kernel is contained in K).

h) If $[G : H] = 2$ then H is normal.

i) Show that if $[G : H] = n$ then $g^{n!} \in H$ for all $g \in G$. If H is normal then $n!$ can be replaced by n. Show that without normality this is no longer true.

Problem 3. Let G be the set of all bijections $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ which preserve distance, i.e. such that $|f(i) - f(j)| = |i - j|$ for all integers i, j.

a) Show that G is a subgroup of $\mathrm{Sym}(\mathbb{Z})$. It is called the **infinite dihedral group** and it is often denoted by D_∞.

b) The group G contains elements T, S such that $T(a) = a + 1$ and $S(a) = -a$ for all integers a. Prove that $S * T = T^{-1} * S$. Show that the subgroup $< T >$ is infinite. What is $< S >$?

c) Show that if $F \in G$ and $F(0) = 0$ then either $F = 1$ (the identity) or $F = S$.

d) Show that every element of G is of the fo T^i or ST^i for some integer i (try to use similar argument to the one we used for dihedral group of order n).

e) Suppose that $T^5 S^7 T^3 = S^a T^b$. Find a and b.

f) Find the center and the derived subgroup of G.

Problem 4. a) Describe all subgroups and normal subgroups of D_n.

b) Describe the center and the derived group of D_n.

c) For which m, n is there a surjective homomorphism from D_m to D_n ? (**Optional:** Describe all homomorphisms from D_m to D_n.)

d) Prove that if x, y are two elements of order 2 in a group G and $xy \neq yx$ then the subgroup $< x, y >$ of G is isomorphic to a dihedral group (finite or infinite).

Problem 5. In the group $GL_2(\mathbb{C})$ of all invertible 2×2 matrices with entries in complex numbers consider the matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$, $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $k = ij = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$. Let Q_8 be the set $\{I, -I, i, -i, j, -j, k, -k\}$.

a) Show that Q_8 is a subgroup of $GL_2(\mathbb{C})$. Write the table of multiplication in Q_8. Q_8 is called the **quaternion group**.

b) List all subgroups of Q_8.

Furthermore, solve problems 18 , 23 to 1.6, problem 6 to 2.1, problem 26 to 2.3 from Dummit and Foote.