Homework 2, part 2

due on Wednesday, March 13

Solve the following problems.

Problem 1. Let z be a complex number such that $|z| \geq 2$. In the group $GL_2(\mathbb{C})$ of all 2×2 invertible matrices with complex entries consider the matrices $a = E_{1,2}(z)$ and $b = E_{2,1}(z)$ (see problem 6 of homework 2). Prove that the subgroup generated by $\{a, b\}$ is a free group on the set $\{a, b\}$.

Problem 2. We consider the group $G = SL_2(\mathbb{Z})$. From Problem 6 in homework 2, G is generated by the two matrices $a = E_{1,2}(1)$ and $b = E_{2,1}(1)$. We proved in class that the subgroup H generated by $\{a^2, b^2\}$ is a free group on the set $\{a^2, b^2\}$.

a) Show that H is not normal in G. Hint: Consider $a^{-1}(a^2b^{-2})a$.

b) Prove that $K = H \cdot \{I, -I\}$ is a normal subgroup of G and G/K is either abelian or dihedral.

c) Let $x = ab^{-1}$ and $y = ab^{-1}a$. Prove that $\{x, y\}$ generates G. What are the orders of x and y? Conclude that G/K is isomorphic to D_3 - the dihedral group of order 6. What is $[G : H]$?

d) Consider the natural homomorphism $\phi : SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/2\mathbb{Z})$. Prove that ϕ surjective and K is the kernel of ϕ.