Homework 2

due on Wednesday, February 28

Read carefully Chapter 2 in Milne's book. Solve problems 2-3, 2-7, 2-9. Read carefully sections 5.1, 5.2, 6.3 in Dummit & Foote. Solve problem 36 to section 3.1, problem 10 to section 4.1, problems 3, 8 to section 5.4.

Solve the following problems.

Problem 1. Let H be a subgroup of $\mathbb{Z}^4$ generated by $(-1,-2,-3,-4)$, $(3,8,5,6)$, $(-1,0,-13,-16)$, $(-3,-4,-13,-6)$. Find a compatible bases of $\mathbb{Z}^4$ and H, then find the rank, the invariant factors, and the elementary divisors of the group $\mathbb{Z}^4/H$.

Problem 2. Let n be a positive integer. An element a in an abelian group A is called n-divisible if $a = nb$ for some $b \in A$. Let p be a prime. We say that a has infinite p-height if a is divisible by every power of p.

Consider the group $P = \prod_{i=1}^{\infty} < x_i >$, where x_i has order p^i. Let T be the torsion subgroup of P. Prove that no non-zero element of P has infinite height. Let $a = (a_i)$ be the element of P with $a_{2i} = x_{2i}^{p^i}$ and $a_{2i-1} = 0$ for $i = 1, 2, \ldots$. Prove that a is not in T and that $a + T$ has infinite p-height in P/T. Conclude that there is no subgroup B in P such that $P = T \oplus B$.

Problem 3. a) Let K, M, N be groups such that K is finite and $K \times M$ is isomorphic to $K \times N$. Prove that M and N are isomorphic.

Hint. Use induction on the order of K. Assume that $G = K_1H_1 = K_2H_2$, where K_1, H_1, K_2, H_2 are normal subgroups of G such that $K_1 \cap H_1 = \{e\} = K_2 \cap H_2$, K_1 and K_2 are both isomorphic to K, H_1 is isomorphic to M and H_2 is isomorphic to N. Consider $T_1 = K_1 \cap H_2$ and $T_2 = K_2 \cap H_1$. Prove that $K_1/T_1 \times K_2/T_2 \times H_1$ and $K_1/T_1 \times K_2/T_2 \times H_2$ are isomorphic.

b) Show that there are infinite groups K for which part a) is false.

Problem 4. For finite groups G, H define $h(G, H)$ to be the number of homomorphisms from G to H and $e(G, H)$ to be the number of injective homomorphisms from G to H.

a) Prove that $h(G, H) = \sum_N e(G/N, H)$, where the sum is over all normal subgroups N of G.

b) Suppose that H_1, H_2 are finite groups such that $h(G, H_1) = h(G, H_2)$ for every finite group G. Prove that H_1 and H_2 are isomorphic. Hint: Show $e(G, H_1) = e(G, H_2)$ for every finite group G.

c) Use b) to give a different proof of a) in problem 3 when all groups are finite. Hint: in class we showed that there is a canonical bijection bitween $\mathrm{Hom}(G, S \times T)$ and $\mathrm{Hom}(G, S) \times \mathrm{Hom}(G, T)$

Problem 5. a) Let $m_1, \ldots, m_n$ be integers whose greatest common divisor is 1. Prove that $\mathbb{Z}^n$ has a basis whose first element is $(m_1, \ldots, m_n)$. Conclude that there is a matrix in $SL_n(\mathbb{Z})$ whose first row is $(m_1, \ldots, m_n)$.

b) Let $w_1, \ldots, w_n$ generate an abelian group A and let $M = (m_{i,j})$ be a matrix in $GL_n(\mathbb{Z})$. Show that the elements $u_i = \sum_j m_{i,j} w_j$ also generate A (compare to Lemma 1.53 in Miln's book).

Problem 6. Let $i \neq j$. An elementary matrix $E_{i,j}(k)$ is a square matrix which has all diagonal entries equal to 1, the i,j-entry equal to k, and all other entries equal to 0 (note that we do not specify the size of the matrix; it will follow from the context). Consider an $m \times n$ matrix $M \neq 0$ with integer entries.

a) Show that $E_{i,j}(s)E_{i,j}(t) = E_{i,j}(s+t)$.

b) Prove that $E_{i,j}(k)M$ is obtained from M be adding k times the j-th row of M to the $i-$th row of M. Formulate and prove a similar statement for $ME_{i,j}(k)$.

c) Let $i \neq j$. Show that there is a product of elementary matrices U such that UM is obtained from M by replacing the i-th row of M with the j-th row of M and the j-th row of M with the negative of the i-th row of M.

d) Let $i \neq j$. Show that there is a product of elementary matrices D such that DM is obtained from M by multiplying both the i-th and j-th rows of M by -1.

e) Explain that in class we proved that there are products of elementary matrices A, B and $s \leq \min(m,n)$ such that AMB is a matrix $(a_{i,j})$ such that the only non-zero entries are $a_{1,1}, a_{2,2}, \ldots, a_{s,s}$ and $a_{1,1}|a_{2,2}|\ldots|a_{s,s}$ and $a_{2,2}, \ldots, a_{s,s}$ are positive.

f) Use e) to show that every matrix in $SL_n(\mathbb{Z})$ is a product of elementary matrices. Conclude that $SL_n(\mathbb{Z})$ is finitely generated.

Remark. a), b), c), d) are true for matrices over any ring. For e) apply the method we used in class to prove the result about compatible bases for a free abelian group of finite rank and its subgroup. First part of f) is also true for matrices over any field

Problem 7 Observe that if A, C are abelian groups then the set $\mathrm{Hom}(A,C)$ of all group homomorphisms from A to C is also an abelian group with addition defined by $(f+g)(a) = f(a)+g(a))$ (verify this).

a) Let A, B, C be abelian groups. Show that the group $\mathrm{Hom}(A \times B, C)$ is naturally isomorphic to $\mathrm{Hom}(A,C) \times \mathrm{Hom}(B,C)$.

b) Suppose that A is a cyclic group of order which divides n and C is a cyclic group of order n. Prove that the groups $\mathrm{Hom}(A,C)$ and A are isomorphic.

c) Use the structure theorem for finite abelian groups to show that if A is abelian of exponent dividing n and C is cyclic of order n then the groups $\mathrm{Hom}(A,C)$ and A are isomorphic.

d) Let A, B, C be abelian groups. A map $F : A \times B \longrightarrow C$ is called **bilinear** if for any $a \in A$ the map $F(a,-) : b \mapsto F(a,b)$ is a homomorphism from B to C and for any $b \in B$ the map $F(-,b) : a \mapsto F(a,b)$ is a homomorphism from A to C. Show that if F is bilinear then the assignment $a \mapsto F(a,-)$ defines a homomorphism from A to $\mathrm{Hom}(B,C)$ and the assignment $b \mapsto F(-,b)$ defines a homomorphism from B to $\mathrm{Hom}(A,C)$. We say that F is **non-degenerate** if these two homomorphisms are injective.

d) Let C be a cyclic group of order n. Suppose that A, B are finite abelian groups and $F : A \times B \longrightarrow C$ is a non-degenrate bilinear map (we ofetn say that F is a non-degenerate pairing in this situation). Prove that the groups A and $\mathrm{Hom}(B,C)$ are isomorphic. Conclude that A and B are isomorphic and of exponent which divides n.

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Challenge. Let G be the product of countable many copies of $\mathbb{Z}$, i.e. $G = \Pi_{i=1}^{\infty} A_i$, where $A_i = \mathbb{Z}$ for all i. Let H be the direct sum of these groups.

a) Prove that if $\phi : G \longrightarrow \mathbb{Z}$ is a homomorphism such that $H < \ker \phi$, then $\ker \phi = G$.

b) Prove that G is not isomorphic to a direct sum of the form $\bigoplus_{i \in I} \mathbb{Z}$. Hint: Show that I is uncountable and try to contradict a).

c) An abelian group without elements of finite order B is called **slender**, if every homomorphisms $\psi : G \longrightarrow B$ maps all but a finite number of A_i to the identity of B. Prove that $\mathbb{Z}$ is slender.

d) Prove that there is no epimorphism of G onto H. Hint: Use c).