Homework 4

due on Friday, April 12

Solve problem 18 to section 4.4, 46 to 4.5, 10, 21 to 5.5, and 21, 24, 26 to 6.1.

Solve the following problems.

Problem 1. Let p be the smallest prime divisor of the order of a finite group G. Prove that if $H < G$ and $[G : H] = p$ then H is normal in G.

Problem 2. Let P be a Sylow p-subgroup of G and let $H < G$. Consider the natural action of H on the left cosets of P in G ($h(aP) = (ha)P$). Show that the stabilizer of some left coset is a Sylow p-subgroup of H. (This result can be used to give a different proof of existence of Sylow p-subgroups).

Problem 3. Prove that there are no simple groups of order 2025.

Problem 4. a) Prove that S_n is generated by $(1, 2)$ and $(1, 2, \ldots, n)$.

b) Show that S_n is generated by $(1, 2), (2, 3), \ldots, (n - 1, n)$.

c) Show that if p is prime then S_p is generated by any set $\{a, b\}$, where a is a transposition and b has order p.

d) Let $d|n$ and $1 < d < n$. Show that S_n is NOT generated by $(1, 2, \ldots, n)$ and $(1, d + 1)$

Problem 5. a) Prove that if P is a Sylow p-subgroup of G and N is normal in G then $N \cap P$ is a Sylow p-subgroup of N.

b) If $f : G \longrightarrow H$ is a surjective homomorphism and P is a Sylow p-subgroup of G then $f(P)$ is a Sylow p-subgroup of H.

Problem 6. Let P be a Sylow p-subgroup of G and M a subgroup of G which contains the normalizer $N_G(P)$. Prove that $N_G(M) = M$. Consider the conjugation action of P on the set of all subgroups conjugate in G to M. Show that M is the only fixed point. Conclude that $[G : M] \equiv 1(\mod p)$.