Homework 5
due on Friday, May 3

Solve problems 18, 32 to 6.1 and problem 5 to 6.2 in Dummit and Foote.

Solve the following problems.

Problem 1. Let R be a commutative ring. Denote by $E_n(R)$ the subgroup of $SL_n(R)$ generated by all elementary matrices $E_{i,j}(r)$, where $1 \leq i, j \leq n$, $i \neq j$, $r \in R$.

a) Prove that

$$[E_{i,j}(r), E_{p,q}(s)] = \begin{cases} I, & \text{if } j \neq p \text{ and } i \neq q \\ E_{i,q}(rs), & \text{if } j = p \text{ and } i \neq q \\ E_{p,j}(-sr), & \text{if } j \neq p \text{ and } i = q \\ \text{something}, & \text{if } j = p \text{ and } i = q. \end{cases}$$

b) Prove than if $n \geq 3$ then $E_n(R)$ is a perfect group. Conclude that $SL_n(\mathbb{Z})$ is perfect for $n \geq 3$.

Remark. When R is not commutative, the group $SL_n(R)$ is not defined since the determinant exists only for commutative rings. However, the group $E_n(R)$ as defined above still makes sense (as a subgroup of $GL_n(R)$, which is the group of invertible elements in the ring $M_n(R)$) and the results of the problem still hold.

Problem 2. Recall that any homomorphism of rings $\phi : R \longrightarrow S$ induces homomorphisms of groups $GL_n(R) \longrightarrow GL_n(S)$ and $SL_n(R) \longrightarrow SL_n(S)$ (by applying ϕ to each entry of a matrix).

Prove that the natural homomorphism $SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/n\mathbb{Z})$ is surjective for all $n > 0$. Hint: start with a matrix A in $SL_2(\mathbb{Z}/n\mathbb{Z})$ and show that one can find relatively prime integers a, b such $A = \left(\begin{smallmatrix} a+n\mathbb{Z} & b+n\mathbb{Z} \\ c+n\mathbb{Z} & d+n\mathbb{Z} \end{smallmatrix}\right)$. Then show that one can choose c, d so that $ad - bc = 1$.

Remark. It is a bit harder, but true, that the natural homomorphism $SL_k(\mathbb{Z}) \longrightarrow SL_k(\mathbb{Z}/n\mathbb{Z})$ is surjective for all k.

Problem 3. Let B be the commutator subgroup $B = [SL_2(\mathbb{Z}), SL_2(\mathbb{Z})]$.

a) Show that $SL_2(\mathbb{Z})/B$ is cyclic of order which divides 12. Hint: Use part c) of Problem 2 from part 2 of homework 2.

b) Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = E_{1,2}(1) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Show that S and T generate $SL_2(\mathbb{Z})$ (use problem 2 from homework 3). Let $u = [S, T] = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$ and $w = [S, T^{-1}] = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ (we use the convention $[a, b] = a^{-1}b^{-1}ab$). Let V be the subgroup of B generated by u, w. Verify that

$$SuS^{-1} = u^{-1}, \quad SwS^{-1} = w^{-1}, \quad TuT^{-1} = w^{-1}, \quad TwT^{-1} = uw.$$

Conclude that $SVS^{-1} = V$ and $TVT^{-1} = V$. Show that V is a normal subgroup of $SL_2(\mathbb{Z})$ and conclude that $V = B$.

c) Let $\overline{u}, \overline{w}$ be the images of u, w in $SL_2(\mathbb{Z}/3\mathbb{Z})$. Show that $\overline{u}^2 = -I = \overline{w}^2$ and $\overline{uw} = -\overline{wu}$. Conclude that the commutator of $SL_2(\mathbb{Z}/3\mathbb{Z})$, which is the image $\overline{B}$ of B, is isomorphic to the quaternion group. Prove that $SL_2(\mathbb{Z}/3\mathbb{Z}) = \overline{B} < \overline{T} > \cong Q_8 \rtimes \mathbb{Z}/3\mathbb{Z}$. Conclude that $PSL_2(\mathbb{Z}/3\mathbb{Z}) \cong A_4$.

d) Let $\overline{u}, \overline{w}$ be the images of u, w in $SL_2(\mathbb{Z}/4\mathbb{Z})$. Show that $\overline{u}\ \overline{w}^{-1}$ and $\overline{u}^{-1}\overline{w}$ generate a normal subgroup K of $\overline{B}$ isomorphic to the Klein 4 group. Conclude that $\overline{B} = K < \overline{u} >$ is isomorphic to A_4. Conclude that $SL_2(\mathbb{Z}/4\mathbb{Z}) = \overline{B} < \overline{T} >$ is isomorphic to a semidirect product $A_4 \rtimes \mathbb{Z}/4\mathbb{Z}$.

e) Use c) and d) to show $SL_2(\mathbb{Z})$ has an epimorphism onto $\mathbb{Z}/3\mathbb{Z}$ and another one onto $\mathbb{Z}/4\mathbb{Z}$. Prove that the index of B in $SL_2(\mathbb{Z})$ is equal to 12.

Problem 4. Let R be a commutative ring. Denote by $UT_n(R)$ the set of all elements in $SL_n(R)$ which are upper triangular matrices with all entries on the main diagonal equal to 1. For $j = 1, 2, \ldots n-1$ define $UT_n^j(R)$ to be the set of all those matrices in $UT_n(R)$ whose entries in the first $j-1$ diagonals above the main diagonal are all 0 (so, in particular, $UT_n^1(R) = UT_n(R)$).

a) Prove that $UT_n^j(R)$ is a subgroup of $SL_n(R)$.

b) Prove that $UT_n^j(R)$ is a normal subgroup of $UT_n(R)$.

c) Prove that $UT_n^j(R)/UT_n^{j+1}(R)$ is isomorphic to the direct sum R^{n-j} of $n-j$ copies of the additive group R.

d) Prove that $[UT_n^j(R), UT_n^i(R)] \leq UT_n^{i+j}(R)$.

e) Prove that if R is a finite field of characteristic p then $UT_n(R)$ is a Sylow p-subgroup of $SL_n(R)$.

Hint. The following way of thinking about the groups $UT_n^j(R)$ may be useful. Let V_i be the subgroup of R^n which consists of all elements $(a_1, \ldots, a_n)$ such that $a_j = 0$ for $j > i$. Thus $V_0 = \{0\} \subseteq V_1 \subseteq \ldots \subseteq V_n = R^n$. Then $UT_n^j(R)$ consists of all those matrices $A \in UT_n(R)$ such that $Av - v \in V_{i-j}$ for all $v \in V_i$ and all i (in other words, A acts trivially on V_i/V_{i-j} for all i; we set $V_k = \{0\}$ for $k \leq 0$).

Problem 5. Let G be a finite group and N a normal Hall subgroup of G. The Schur-Zassenhaus theorem states that N has a complement in G and any two complements are conjugate. We proved in class that a complement exists. Moreover, we showed that any two complements are conjugate if N is abelian.

a) Using the the fact that complements are conjugate if N is an elementary abelian p-group as a starting point, prove that any two complements are conjugate if N is solvable (use induction on $|N|$.

b) Use induction on $|G/N|$ to prove that any two complements of N are conjugate if G/N is solvable.

c) Explain why the odd order theorem and parts a), b) imply the general case (that any two complements of N are conjugate).

Remark. It is an open problem to give a simpler proof of this result which does not use the odd order theorem.

Problem 6. Let G be a finite group and let H, L be two Hall subgroups of G of the same order.

a) Prove that $N_G(H) \cap L = H \cap L$, where $N_G(H)$ is the normalizer of H in G.

b) Prove that $N_G(N_G(H)) = N_G(H)$.

c) Suppose that G is solvable and M contains $N_G(H)$. Prove that $M = N_G(M)$.