Definition: Let $\mathcal{F}$ be a class of groups. We ~~also~~ say that a group G is poly-$\mathcal{F}$ if it has a subnormal series

$$\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G$$

such that $G_i/G_{i-1} \in \mathcal{F}$ for $i = 1, 2, \ldots, n$.

Remark: We assume that if a group H is in $\mathcal{F}$ then any group isomorphic to H is also in $\mathcal{F}$ (i.e. $\mathcal{F}$ is a class of isomorphisms types of groups).

Examples:

(1) $\mathcal{F}$ = all finite groups. Then poly-$\mathcal{F}$ = all finite groups (if each G_i/G_{i-1} is finite then G is finite).

(2) $\mathcal{F}$ = all cyclic groups. Then groups in poly-$\mathcal{F}$ are called polycyclic groups. This is an important class of groups

(3) $\mathcal{F}$ = all abelian groups. Then groups in poly-$\mathcal{F}$ are called solvable groups.

(4) $\mathcal{F}$ = all simple groups. Then groups in poly-$\mathcal{F}$ are exactly all the groups which have a composition series.

(5) $\mathcal{F} = \{\mathbb{Z}/p\mathbb{Z}, \{e\}\}$. Then poly-$\mathcal{F}$ = all finite p-groups (here p is a prime). Clearly every poly-$\mathcal{F}$ group is a finite p-group. We will show the converse a bit later.

(6) poly-(poly-$\mathcal{F}$) = poly-$\mathcal{F}$

Proposition 1: Let G be a group and $N \trianglelefteq G$. If both N and G/N are poly-$\mathcal{F}$ then G is poly $\mathcal{F}$.

Proof: Since N is poly-$\mathcal{F}$, we have a subnormal series:

$$\{e\} = N_0 \leq N_1 \leq \dots \leq N_k = N \text{ s.t. } N_i/N_{i-1} \in \mathcal{F}.$$

Since G/N is poly-$\mathcal{F}$, we have a subnormal series:

$$\{e\} = G_0/N \leq G_1/N \leq \dots \leq G_m/N = G/N \text{ where } {}^{G_i/N}\!/_{G_{i-1}/N} \in \mathcal{F}$$

and $G_0 = N \leq G_1 \leq \dots \leq G_m = G$, $G_{i-1} \triangleleft G_i$ for all i (by the correspondence theorem). Note that ${}^{G_i/N}\!/_{G_{i-1}/N} \simeq G_i/G_{i-1} \in \mathcal{F}$.

Thus $\{e\} = N_0 \leq N_1 \leq \dots \leq N_k = N = G_0 \leq G_1 \leq \dots \leq G_m = G$ is a subnormal series in G and each successive quotient is in $\mathcal{F}$.

This shows that G is poly-$\mathcal{F}$. $\boxtimes$

Proposition 2: If $\mathcal{F}$ is closed under subgroups (i.e. if $H \in \mathcal{F}$ and $K < H$ then $K \in \mathcal{F}$) and $G \in$ poly-$\mathcal{F}$ then every subgroup of G is also poly-$\mathcal{F}$.

Proof: Since G is poly-$\mathcal{F}$, we have a subnormal series:

$$\{e\} = G_0 \leq G_1 \leq G_2 \leq \dots \leq G_n = G \text{ with } G_i/G_{i-1} \in \mathcal{F} \text{ for all } i.$$

Let H be a subgroup of G. Then $H_i \overset{\text{def}}{=} G_i \cap H$ are subgroups of H and $H_0 = \{e\} \leq H_1 \leq \dots \leq H_n = G_n \cap H = G \cap H = H$. Since $G_{i-1} \triangleleft G_i$ we have $H_{i-1} = G_{i-1} \cap H \triangleleft G_i \cap H$. Also ${}^{G_i \cap H}\!/_{G_{i-1} \cap H}$ is isomorphic to a subgroup of G_i/G_{i-1}: the natural map $a(G_{i-1} \cap H) \longrightarrow a \cdot G_{i-1}$ is injective group homomorphism from ${}^{G_i \cap H}\!/_{G_{i-1} \cap H}$ to G_i/G_{i-1}.

Thus $H_0 = \{e\} \leq H_1 \leq \dots \leq H_n = H$ is a subnormal series in H with successive quotient in $\mathcal{F}$ (since $\mathcal{F}$ is closed under subgroups) i.e. H is poly-$\mathcal{F}$. $\boxtimes$

<u>Proposition 3:</u> If $\mathcal{F}$ is closed under quotients (i.e. if $G \in \mathcal{F}$ and $N \trianglelefteq G$ then $G/N \in \mathcal{F}$) and G is poly-$\mathcal{F}$ then every quotient of G is poly-$\mathcal{F}$.

<u>Proof:</u> Since G is poly-$\mathcal{F}$, we have a subnormal series $\{e\} = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$ s.t. $G_i/G_{i-1} \in \mathcal{F}$.

Let $N \trianglelefteq G$. Then NG_i is a subgroup of G and

$$\{e\} = NG_0/N = H_0 \trianglelefteq NG_1/N = H_1 \trianglelefteq \dots \trianglelefteq NG_n/N = H_n = G/N$$

is a subnormal series in G/N. Moreover

$$H_i/H_{i-1} = NG_i/NG_{i-1} \simeq G_i/G_{i-1}(N \cap G_i) \text{ is a quotient of } G_i/G_{i-1}$$

(by $G_{i-1}(N \cap G_i)/G_{i-1}$), so $H_i/H_{i-1} \in \mathcal{F}$. Thus G/N is poly-$\mathcal{F}$. $\boxtimes$

We can now derive several corollaries. Since $\mathcal{F}$ = abelian, $\mathcal{F}$ = cyclic, or $\mathcal{F} = \{\mathbb{Z}/p\mathbb{Z}, \{e\}\}$ are closed under subgroups and quotient groups we get:

① If $N \trianglelefteq G$ then G is solvable iff both N and G/N are solvable.
Any subgroup of a solvable group is solvable.

② If $N \trianglelefteq G$ then G is polycyclic iff both N and G/N are polycyclic. Any subgroup of a polycyclic group is polycyclic.

We can now justify our example 5, i.e. show that any finite p-group is poly-$\{\mathbb{Z}/p\mathbb{Z}, \{e\}\}$. We show it by induction on n, where $|P| = p^n$. If $n=1$, i.e. $|P| = p$ then $P \simeq \mathbb{Z}/p\mathbb{Z}$, so P is poly-$\{\mathbb{Z}/p\mathbb{Z}, \{e\}\}$.
Assume that p-groups of order less than p^n are poly-$\{\mathbb{Z}/p\mathbb{Z}, \{e\}\}$ and consider a p-group P of order p^n. Recall that every finite p-group has a non-trivial center. It follows that

P has a central element a of order p. Let $N = \langle a \rangle$. Then $N \simeq \mathbb{Z}/p\mathbb{Z}$ and $N \trianglelefteq P$. By induction, N and P/N are poly-$\{\mathbb{Z}/p\mathbb{Z}, \{e\}\}$ and therefore P is by Proposition 1. ☒

We are going now to study solvable groups. Recall that G/H is abelian iff $[G,G] \subseteq H$.

<u>Definition</u>: For a group G define $G^{(0)} = G$, $G^{(1)} = [G,G]$ and

$$G^{(i+1)} = [G^{(i)}, G^{(i)}] \text{ for } i \geq 0.$$

The group $G^{(k)}$ is called the k-th derived subgroup of G.

Note that each $G^{(k)}$ is a characteristic subgroup of G.

Also, $G^{(k)}/G^{(k+1)}$ is abelian for all k.

Suppose $\{e\} = G_0 \subseteq G_1 \subseteq \ldots \subseteq G_n = G$ is a subnormal series such that G_i/G_{i-1} is abelian for all i. Then $[G_i, G_i] \subseteq G_{i-1}$ for all i!

<u>Claim</u>: $G^{(k)} \subseteq G_{n-k}$ for $k = 0, 1, \ldots, n$.

In fact, this is true for $k=0$ and if $G^{(k)} \subseteq G_{n-k}$ then

$$G^{(k+1)} = [G^{(k)}, G^{(k)}] \subseteq [G_{n-k}, G_{n-k}] \subseteq G_{n-k-1} = G_{n-(k+1)}$$

so the claim follows by induction.

In particular $G^{(n)} \subseteq G_0$, i.e. $G^{(n)} = \{e\}$.

We see that if G is solvable, then $G^{(n)} = \{e\}$ for some n.

Conversely, if $G^{(n)} = \{e\}$ then $\{e\} = G^{(n)} \subseteq G^{(n-1)} \subseteq \ldots \subseteq G^{(0)} = G$ is a subnormal (actually normal) series with successive quotients abelian. Thus G is solvable. In other words:

<u>Theorem</u>: G is solvable if and only if $G^{(n)} = \{e\}$ for some n.

<u>Def</u>: Let G be solvable. Let k be the smallest integer such that $G^{(k)} = \{e\}$. We call k the <u>derived length of G</u> (or the <u>solvability class of G</u>). From our discussion above: The derived length of G is the shortest possible length of a subnormal series in G with abelian quotients.

<u>Example</u>: Consider $G = GL_2(\mathbb{Q})$. We proved that $SL_2(\mathbb{Q})$ is perfect. Also, every element of $[G,G]$ has determinant 1, so $[G,G] = SL_2(\mathbb{Q})$. In other words, $G^{(1)} = SL_2(\mathbb{Q})$. Since $SL_2(\mathbb{Q})$ is perfect, $G^{(2)} = G^{(3)} = \dots = SL_2(\mathbb{Q})$. Thus G is <u>not solvable</u>.

More generally, if G has a perfect subgroup or a perfect quotient then G can not be solvable.