Homework 2

due on Friday, September 23

Read carefully sections 13.4, 13.5 in the book. Also read the first 28 pages of the on-line book by J.S. Milne.

Solve problems 3,4 to section 13.4, problems 2,5 to section 13.5, problems 4,5,10 to section 14.1 and the following problems:

Problem 1. *Let L/K be an algebraic extension of fields. Suppose that $K \subseteq R \subseteq L$ and R is a subring of L. Prove that R is a field.*

Problem 2. *a) Let $f(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$ be a polynomial in $k[x]$. Prove that*

$$f(x) = \det \begin{pmatrix} x & 0 & \cdots & 0 & 0 & a_0 \\ -1 & x & \cdots & 0 & 0 & a_1 \\ 0 & -1 & \cdots & 0 & 0 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x & a_{k-2} \\ 0 & 0 & \cdots & 0 & -1 & x + a_{k-1} \end{pmatrix}$$

b) Let L/K be a finite extension of degree n. Show that if $a \in L$ and d is the degree of the minimal polynomial of a over K then $d|n$.

c) Let L/K be a finite extension of degree n. Multiplication by $a \in L$ is a linear transformation m_a of L considered as a vector space over K. Let f be the minimal polynomial of a over K, $d = degf$. Prove that the characteristic polynomial of m_a equals $f^{n/d}$.

Hint. *Consider a a basis of L/K which is built from the basis $1, a, ..., a^{d-1}$ of $K[a]/K$ and some basis of $L/K[a]$ and use a).*

d) Define two maps $T_{L/K}$ and $N_{L/K}$ from L to K as follows: $T_{L/K}(a)$ is the trace of m_a and $N_{L/K}$ is the determinant of m_a. Show that $N_{L/K}$ is a group homomorphism of multiplicative groups: $N_{L/K} : L^\times \longrightarrow K^\times$. Show that $T_{L/K} : L \longrightarrow K$ is a homomorphism of $K-$vector spaces. $T_{L/K}$ is called the **trace** *from L to K and $N_{L/K}$ is the* **norm** *form L to K.*

e) Show that if $[L : K] = n$ and $a \in L$ has minimal polynomial $f(x) = x^d + a_{d-1}x^{d-1} + ... + a_0$ then $N_{L/K}(a) = (-1)^d a_0^{n/d}$ and $T_{L/K}(a) = -na_{d-1}/d$.

f) Let $K \subseteq L \subseteq M$ be fields, $[M : K] < \infty$. Prove that $T_{L/K} \circ T_{M/L} = T_{M/K}$ and $N_{L/K} \circ N_{M/L} = N_{M/K}$

g) Show that if F is a finite field with q elements and L/F is a finite extension of degree n then $T_{L/F}(a) = a + a^q + ... + a^{q^{n-1}}$ and $N_{L/F}(a) = a^{1+q+q^2+...+q^{n-1}}$. Conclude that both $T_{L/F}$ and $N_{L/F}$ are surjective.