Midterm

due on Monday, November 7

Problem 1. Let K be a field and let p be a prime number not equal to the characteristic of K. Suppose that K contains primitive $p-$th root of 1 and if $p = 2$ also the primitive 4-th root of 1.

1. Suppose that for some k the splitting fields of $x^{p^k} - 1$ and of $x^{p^{k+1}} - 1$ coincide. Prove that K contains primitive p^{k+1}-th root of 1.
2. Suppose that K contains primitive p^k-th root of 1 for all k. Prove that if for some $a \in K$ the polynomials $x^p - a$ and $x^{p^2} - a$ have the same splitting fields over K then both polynomials have all their roots in K.

Hint: If p is odd and p^{k+1} divides $m^p - 1$ then p^k divides $m - 1$.

Problem 2. Let L/K be a Galois extension. We say that $a \in L$ generates a normal basis of L/K if the set $\{\tau(a) : \tau \in \text{ Gal}(L/K)\}$ is a basis of L over K. Let $K \subseteq M \subseteq L$ be a subfield such that M/K is Galois. Prove that if $a \in L$ generates a normal basis of L/K then the trace $Tr_{L/M}(a)$ generates a normal basis of M/K.

Problem 3. Let $f \in \mathbb{Z}[x]$ be a monic polynomial of degree n with roots x_1,..., x_n.

1. Prove that for every integer $k > 0$ there is a monic polynomial $g_k \in \mathbb{Z}[x]$ of degree n whose roots are $x_1^k, x_2^k, ..., x_n^k$ (one way is to use symmetric functions).
2. Suppose that the absolute values $|x_i|$ satisfy $|x_i| \leq 1$ for all i. Prove that the sequence g_1, g_2,... from 1. contains only a finite number of different polynomials (Hint: bound the coefficients of g_k). Conclude that each x_i is a root of unity.

Problem 4. Consider the polynomial $p(x) = x^4 + 5x^2 + 12x + 13$.

1. Prove that p is irreducible over $\mathbb{Q}$.
2. Find the Galois group of the splitting field of p. Provide all details of your solution.
3. Express the roots of p in radicals.