Homework 1

due on Friday, February 11

Read carefully section I.1 in the book and Appendix A. Read Chapter 1 in Milne's book (on-line). Solve the following problems.

Problem 1. Let K be a field and let a be algebraic over K of odd degree. Prove that $K[a] = K[a^2]$. Show by example that this can be false when the degree of a is even.

Problem 2. Let $f(x)$ be an irreducible polynomial of degree n over a field K. Let $g(x)$ be any polynomial in $K[x]$. Prove that any irreducible factor of $f(g(x))$ over K has degree divisible by n. Hint: Work with roots of f and $f(g(x))$ in some bigger field L and use the notion of degree.

Problem 3. Let K be a field of characteristic not equal to 2. Let $a, b \in K$ be elements which are not suqres in K and let $L = K(\sqrt{a}, \sqrt{b})$.

a) Prove that $[L : K] = 2$ iff $K[\sqrt{a}] = K[\sqrt{b}]$ iff ab is a square in K.

b) Prove that if ab is not a square in K then $[L : K] = 4$ and $L = K(\sqrt{a} + \sqrt{b})$.

Problem 4. Let s_1,..., s_n be the elementary symmetric polynomials in the variables $X_1, ..., X_n$. Let $p_k = X_1^k + \ldots + X_n^k$, $k = 1, 2, \ldots$. Prove the following formulas (called **Newton's formulas**):

$$p_i - s_1 p_{i-1} + s_2 p_{i-2} - \ldots + (-1)^{i-1} s_{i-1} p_1 + (-1)^i i s_i = 0$$

for $i = 1, 2, \ldots, n$, and

$$p_i - s_1 p_{i-1} + s_2 p_{i-2} - \ldots + (-1)^{i-1} s_{n-1} p_{i-n+1} + (-1)^i s_n p_{i-n} = 0$$

for all $i \geq n$.

Problem 5. Let K be a field of characteristic not equal to 2. Let a, b be elements of K such that b is not a square in K. Let $u = \sqrt{a + \sqrt{b}}$ (i.e. u is a root of $(x^2 - a)^2 - b$). Prove that $K(u)$ is of the form $K(\sqrt{m}, \sqrt{n})$ for some $m, n \in K$ if and only if $a^2 - b$ is a square in K. Prove furthermore that m, n can be chosen so that $u = \sqrt{m} + \sqrt{n}$. Find rational numbers m, n such that $\sqrt{3 + \sqrt{5}} = \sqrt{m} + \sqrt{n}$.