Homework 4

due on Thursday, April 11

Solve the following problems.

Problem 1. Solve problem 22 to section 14.6 in Dummit and Foote (about Newton's formulas).

Problem 2. Solve problem 27 to section 14.6 in Dummit and Foote (in part a) justify the formula for the determinant).

Problem 3. Consider the polynomial $p(x) = x^4 + 5x^2 + 12x + 13$.

a) Prove that p is irreducible over $\mathbb{Q}$. Compute the discriminant of p.

b) Let x_1, x_2, x_3, x_4 be the roots of p. Let $z_1 = x_1x_2 + x_3x_4$, $z_2 = x_1x_3 + x_2x_4$ and $z_3 = x_1x_4 + x_2x_3$. Let $q(x) = (x - z_1)(x - z_2)(x - z_3)$. Explain why q should have rational coefficients and compute these coefficients. Then find the roots of q.

c) Consider the Galois group G of p as a subgroup of S_4 via its permutation action on the roots of p. Prove that $\mathbb{Q}(x_1, x_2, x_3, x_4)/Q(z_1, z_2, z_3)$ is Galois with Galois group $G \cap V$, where V is the unique normal subgroup of S_4 of order 4. Conclude that the Galois group of p is contained in a Sylow 2-subgroup of S_4. Prove that $V \subseteq G$ and conclude that G is isomorphic to the dihedral group of order 8 (one way to do that is to show that $Q(x_1, x_2, x_3, x_4)$ contains two quadratic extensions of $\mathbb{Q}$).

c) Express the roots of p in radicals.

Problem 4. Solve problem 3 to section 14.9 in Dummit and Foote.

Problem 5. Let K ba a field and p a prime number not equal to the characteristic of K. Suppose that K contains a primitive pth root of 1 and, if $p = 2$, also the promitive 4-th root of 1.

a) Suppose that for some k the splitting fields of $x^{p^k} - 1$ and $x^{p^{k+1}} - 1$ over K coincide. Prove that both splliting fields are equal to K. Hint: prove first that if p is odd and p^{k+1} divides $m^p - 1$ for some integer m then p^k divides $m - 1$.

b) Suppose that for some $a \in K$ the splitting fields of $x^p - a$ and $x^{p^2} - a$ over K coincide. Prove that both splliting fields are equal to K.