Homework 1

due on Friday, September 18

Problem 1. Let U, W be subspaces of a vector space V such that $U \cup W$ is a subspace. Prove that either $U \subseteq W$ or $W \subseteq U$.

Problem 2. a) Prove that the vector space $C(\mathbb{R})$ of all continuous functions is not finitely generated.

b) Show that the function $\sin 2x$ is not a linear combination of $\sin x$ and $\cos x$.

Problem 3. Prove that the commutativity of addition is a consequence of the other axioms in the definition of a vector space.

Problem 4. Let K be the set of all real numbers of the form $a + b\sqrt{5}$, where a, b are rational numbers. Prove that this set is a field under the ordinary addition and multiplication of numbers.

Problem 5. Let V be the set of positive real numbers. Define the operation $+$ on V as $v + w = vw$ (the usual product of numbers). For $a \in \mathbb{R}$ and $v \in V$ define the scalar multiplication as $a \cdot v = v^a$ (the usual exponentiation of numbers). Finally choose $1 \in V$ for the distinguished element. Prove that V is a vector space over the real numbers under the above operations.

Problem 6. Find two subsets A, B of $\mathbb{R}^3$ such that $\text{span}A \cap \text{span}B \neq \text{span}(A \cap B)$.

Problem 7. Let v, w be two elements of a vector space V over a field F. Consider the set $L = \{tv + (1-t)w : t \in F\}$. Prove that there is a vector $u \in V$ such that $u + L$ is a subspace of V.