Homework 2

due on Wednesday, September 29

Problem 1. Let W be a subspace of a finite dimensional vector space V. Prove that $\dim V/W = \dim V - \dim W$.

Problem 2. Prove that a vector space V is not finite dimensional iff it contains an infinite linearly independent subset.

Problem 3. Let v_1, v_2,...,v_n be a basis of a vector space V over a field K. When is $v_1 + v_2$, $v_2 + v_3$,..., $v_n + v_1$ a basis of V?

Problem 4. a) Find the reduced row-echelon form of the matrix

$$\begin{pmatrix} 0 & 2 & 4 & 2 & 2 \\ 4 & 4 & 4 & 8 & 0 \\ 8 & 2 & 0 & 10 & 2 \\ 6 & 3 & 2 & 9 & 1 \end{pmatrix}$$

List the elementary row operations performed. What is the rank of this matrix?

b) Find the dimension and a basis of the subspace of $\mathbb{R}^5$ spanned by the vectors $(1,1,0,-1,-1)$, $(1,0,-1,0,0)$, $(1,2,1,-2,-2)$, $(2,1,1,1,2)$, $(4,3,-1,-3,-3)$. Express all these vectors as linear combination of vectors in the constructed basis.

Problem 5. a) Solve the system of linear equations:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + 4x_4 - 9x_5 &= 17 \\ x_1 + x_2 + x_3 + x_4 - 3x_5 &= 6 \\ x_1 + x_2 + x_3 + 2x_4 - 5x_5 &= 8 \\ 2x_1 + 2x_2 + 2x_3 + 3x_4 - 8x_5 &= 14 \end{aligned}$$

by finding a basis of the space of solutions of the associated homogeneous system and a solution to the given system. Verify your answer by checking that the vectors really are solutions.

b) Find a system of linear homogeneous equations whose solution space is spanned by the vectors from problem 4b). Verify your answer. What is the minimal possible number of equations in such a system ?

Problem 6. Let $\mathbf{v}_1 = (1,0,-1,0)$, $\mathbf{v}_2 = (1,-1,0,0)$, $\mathbf{v}_3 = (1,0,0,-1)$. Set $V = \text{span}(\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3,\})$. Let $\mathbf{w}_1 = (1,1,1,1)$, $\mathbf{w}_2 = (1,2,1,0)$, $\mathbf{w}_3 = (0,1,2,1)$. Set $W = \text{span}(\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\})$. Find a basis of $V \cap W$. What can you say about $V + W$?.

Problem 7. Let V be a finite dimensional vector space over an infinite field K. Given proper subspaces W_1, W_2,..., W_k of V prove that there is a vector in V which does not belong to any of these subspaces.