Homework 3

due on Wednesday, October 13

Problem 1. A matrix $A = (a_{ij}) \in M_n(\mathbb{C})$ satisfies

$$|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1, 2, ..., n.$$

Prove that A is invertible. (Hint: Consider the system of homogeneous linear equations with A as coefficient matrix).

Problem 2. Let A be an $m \times n$ matrix.

a) Prove that there exist elementary matrices M_1,...,M_k of type $E_{i,j}(a)$ such that for some $u \neq 0$ the matrix $S_1(u)M_1...M_kA$ is the reduced row echelon form of A.
Hint Prove that given i, j, k, u, a, one can write $E_{i,j}(a)S_k(u) = S_k(u)E_{i,j}(w)$ for some w. Then prove that given k and t there is u such that $S_1(u)S_k(t)$ is a product of elementary matrices of type $E_{i,j}(a)$ for any $k \neq 1$.

b) Prove that if the reduced row echelon form of A has a zero row (i.e. its rank is smaller than m) then one can take $u = 1$ in a).

c) Is the 3×3 matrix $T_{1,3}$ a product of elementary matrices of type $E_{i,j}(a)$?

Problem 3. Find the inverse of

$$a) \quad A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}, \qquad b) \quad A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Verify your answer. Express A as a product of elementary matrices.

Problem 4. Let A, B be square matrices of the same size n such that $AB = 0$. Prove that $\text{rank}(A) + \text{rank}(B) \leq n$.

Problem 5. Consider a system of linear equations $Ax = b$, where both A and b have entries in the field of rational numbers. Prove that if this system has a solution in real numbers then it also has a solution in rational numbers.

Problem 6. Let A ba an $m \times n$ matrix and B an $n \times k$ matrix. Prove that

$$\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}.$$