Homework 4

due on Wednesday, October 20

Problem 1. Compute the determinant of

$$a)\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}\quad b)\begin{pmatrix}1&-1&2&-1\\-3&4&1&-1\\2&-5&-3&8\\-2&6&-4&1\end{pmatrix}\quad c)\begin{pmatrix}1&1&0&0&0\\-1&1&1&0&0\\0&-1&1&1&0\\0&0&-1&1&1\\0&0&0&-1&1\end{pmatrix}$$

Problem 2. Let $a, b, p_1, ..., p_n \in K$. Consider the matrix

$$A=\begin{pmatrix}p_1&a&a&a&\dots&a&a\\b&p_2&a&a&\dots&a&a\\b&b&p_3&a&\dots&a&a\\b&b&b&p_4&\dots&a&a\\\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\b&b&b&b&\dots&p_{n-1}&a\\b&b&b&b&\dots&b&p_n\end{pmatrix}.$$

Set $f(x) = (p_1 - x)(p_2 - x)...(p_n - x)$ and $f_i(x) = f(x)/(p_i - x)$.

a) Show that, if $a \neq b$, then $\det A = \dfrac{bf(a) - af(b)}{b - a}$.

b) Show that, if $a = b$, then $\det A = a\sum_{i=1}^{n} f_i(a) + p_n f_n(a)$.

c) Evaluate the determinant of the matrix

$$\begin{pmatrix}b&a&a&a&\dots&a&a\\a&b&a&a&\dots&a&a\\a&a&b&a&\dots&a&a\\a&a&a&b&\dots&a&a\\\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\a&a&a&a&\dots&b&a\\a&a&a&a&\dots&a&b\end{pmatrix}.$$

Problem 3. Show that

$$\det\begin{pmatrix}1&1&1&\dots&1\\a_1&a_2&a_3&\dots&a_n\\a_1^2&a_2^2&a_3^2&\dots&a_n^2\\\vdots&\vdots&\vdots&\dots&\vdots\\a_1^{n-1}&a_2^{n-1}&a_3^{n-1}&\dots&a_n^{n-1}\end{pmatrix}=\Pi_{1\leq i<j\leq n}(a_j-a_i).$$

Problem 4. Show that if $a \neq b$ then

$$\det \begin{pmatrix} a+b & ab & 0 & \dots & 0 & 0 \\ 1 & a+b & ab & \dots & 0 & 0 \\ 0 & 1 & a+b & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \\ 0 & 0 & 0 & \dots & a+b & ab \\ 0 & 0 & 0 & \dots & 1 & a+b \end{pmatrix} = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

(this is $n \times n$ matrix). What if $a = b$?

Problem 5. Show that if A is an $n \times n$ matrix with entries 1 and -1 then $\det A$ is divisible by 2^{n-1}.

Problem 6. Let $A, B, C, D \in M_n(\mathbb{R})$. Define a $2n \times 2n$ matrix G by

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

a) Suppose that G has rank n. Prove that

$$\det \begin{pmatrix} \det A & \det B \\ \det C & \det D \end{pmatrix} = 0.$$

Moreover, show that if A is invertible, then $D = CA^{-1}B$.

b) Show that if A is invertible then $\det G = \det A \det(D - CA^{-1}B)$.

c) Show that if $AC = CA$ then $\det G = \det(AD - CB)$. Is this true without the assumption that A and C commute?

Problem 7. a) Show that if A and B are $n \times n$ matrices with integral entries and $a_{i,j} - b_{i,j}$ is even for all i, j then $\det(A) - \det(B)$ is even.

b) Show that the determinant of the $n \times n$ matrix whose diagonal entries are 0 and off diagonal entries are all 1 equals $(-1)^{n-1}(n-1)$.

c) Use a) and b) to show that if n is even and an $n \times n$ matrix A has even diagonal entries and odd off diagonal entries then $\det(A) \neq 0$.

d) Prove that for n odd any matrix A as in c) has rank at least $n-1$.

e) Any 2000 coins from a collection of 2001 coins can be divided into 2 groups of 1000 coins such that each group is worth the same. Prove that all coins have the same value.

f) Any 1999 coins from a collection of 2000 coins can be divided into 2 groups such that each group is worth the same. Prove that all coins are worthless.

Hint: For e) and f) you can either use the previous parts or be clever and first solve the problems assuming that the value of each coin is an integer.