Homework 5

due on Monday, November 1

Problem 1. Let $T : V \longrightarrow W$ be a linear transformation. Choose a basis $u_1, ..., u_k$ of $\ker T$.

a) Complete $u_1, ..., u_k$ to a basis $u_1, ..., u_k, u_{k+1}, ..., u_{k+m}$ of V. Prove that $T(u_{k+1})$,...,$T(u_{k+m})$ is a basis of the image ImT of T. Conclude that $\dim V = \dim \ker T + \dim \mathrm{Im}T$.

b) Let $w_1, ..., w_m$ be a basis of ImT and choose u_{k+i} such that $T(u_{k+i}) = w_i$, $i = 1, ..., m$. Prove that $u_1, ..., u_k, u_{k+1}, ..., u_{k+m}$ is a basis of V.

Problem 2. Let A be a matrix of size 5×3 and let B be a matrix of size 3×5. Prove that the 5×5 matrix AB is not invertible. (Hint: Think in terms of linear transformations).

Problem 3. A linear transformation $S : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ is given by the matrix $A = \begin{pmatrix} 2 & 6 & 1 & 4 & 4 \\ 1 & 3 & 1 & 3 & 3 \\ 1 & 3 & 2 & 5 & 5 \\ 1 & 3 & 3 & 7 & 7 \end{pmatrix}$.

Find bases of the kernel and of the image of S.

Problem 4. a) Find the matrix of the linear transformation

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \qquad T(x, y, z) = (x + y + z, x - y + z)$$

in the basis $(1, -2, 1), (1, 2, 1), (0, 2, 1)$ of $\mathbb{R}^3$ and the basis $(1, 1), (1, -1)$ of $\mathbb{R}^2$.

b) What is the change of basis matrix from the basis $\mathbf{b} = \{(1, 1, 1), (1, 0, 1), (0, 0, 1)\}$ to the basis $\mathbf{d} = \{(2, 1, 1), (2, 2, 1), (3, 2, 2)\}$ of $\mathbb{R}^3$?. Find the coordinates of a vector in the basis $\mathbf{d}$ if its coordinates in the basis $\mathbf{b}$ are $(0, 1, 2)$.

Problem 5. Find a linear transformation $T : \mathbb{R}^6 \longrightarrow \mathbb{R}^4$ such that $\ker T$ has basis $(1, 0, 2, 0, -2, 3)$, $(0, 1, -1, 0, 1, 1)$, $(0, 0, 0, 1, -2, 2)$.

Problem 6. a) A liner transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ satisfies $T(1, 0, 1) = (1, 1, 0)$ and $T(1, 1, 0) = (1, 1, 1)$. What is $T(5, 3, 2)$?

b) Is there a liner transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ which satisfies $T(1, 1) = (1, 1, 0)$, $T(1, -1) = (1, 1, 1)$ and $T(3, 1) = (1, 0, 0)$?

Problem 7. Let $T : V \longrightarrow W$ be a liner transformation.

a) Prove that T is injective iff there is a liner transformation $S : W \longrightarrow V$ such that ST is the identity map on V.

b) Prove that T is surjective iff there is a liner transformation $S : W \longrightarrow V$ such that TS is the identity map on W.

Problem 8. Let $T : V \longrightarrow V$ be a linear transformation. Suppose that $v \in V$ is such that $T^n(v) = 0$ but $T^{n-1}(v) \neq 0$. Prove that $v, T(v), T^2(v), ..., T^{n-1}(v)$ are linearly independent. Here T^k is the composition $T \circ T \circ ... \circ T$ of T with itself $k-$times.

Problem 9. Let $T : V \longrightarrow V$ be a linear transformation such that T and T^2 have the same image. Prove that $V = \ker T \oplus \mathrm{Im}T$.