Homework 6

due on Wednesday, November 17

Problem 1. Let $T : V \longrightarrow V$ be a linear transformation. Suppose that $0 < k < \dim V$ and that every subspace of V of dimension k is T-invariant. Prove that $T = aI$ for some constant a.

Problem 2. a) Let $T : V \longrightarrow V$ be a linear transformation. Prove that $\ker T^i \subseteq \ker T^{i+1}$ and $\mathrm{Im}T^{i+1} \subseteq \mathrm{Im}T^i$ for every non-negative integer i. Prove furthermore that if $\ker T^k$ and $\ker T^{k+1}$ have the same dimension for some integer k then all the kernels $\ker T^i$ have the same dimension for $i \geq k$.

b) Use a) to show that if the matrices A^k and A^{k+1} have the same rank, then all the matrices A^i with $i \geq k$ have the same rank.

Problem 3. Let K be a field.

a) Let $f(x), g(x) \in K[x]$ and $g(x) \neq 0$. Prove that there exist unique polynomials $d(x)$ and $r(x)$ such that $\deg r < \deg g$ and $f = dg + r$. The polynomial r is called the **remainder** upon division of f by g. Find d and r if $f(x) = x^5 - x^3 + 3x - 5$ and $g(x) = x^3 + x - 1$ (use the division algorithm for polynomials).

b) Let $f(x), g(x) \in K[x]$ and $f(x) \neq 0$. Prove that there exists unique monic polynomial $d(x)$ such that

i) $d(x)$ divides both $f(x)$ and $g(x)$;

ii) if $h(x)$ divides both $f(x)$ and $g(x)$ then $h(x)$ divides $d(x)$.

Furthermore, prove that $d(x) = a(x)f(x) + b(x)g(x)$ for some polynomials $a, b \in K[x]$. Conclude that if L is a subfield of K and both f, g belong to $L[x]$ then $d \in L[x]$.

The polynomial d is called the **greatest common divisor** of f and g and it is denoted by $\gcd(f, g)$. Given f, g one can compute d using **Euclidean** algorithm: define $f_0 = g$, $f_1 = f$ and for $n \geq 2$ define f_n as the remainder upon division of f_{n-2} by f_{n-1} if $f_{n-1} \neq 0$ and as 0 otherwise (i.e. if $f_{n-1} = 0$). Then d is the last non-zero member of the sequence f_n divided by its leading coefficient (to make it monic).

c) Find the greatest common divisor of $x^5+7x^4-x^3-13x^2-2x-2$ and $x^4+6x^3-8x^2-12x+12$.

d) Prove that if $f(x) \in K[x]$ has degree at most 3 then f is irreducible iff f has no roots in K.

Problem 4. Let U be a $T-$invariant subspace of V and let S be a subset of V. Prove that the set I of all polynomials f such that $f(T)(v) \in U$ for every $v \in S$ (i.e. $I = \{f : f(T)(v) \in U \text{for all} v \in S\}$) is an ideal. Show that this ideal contains non-zero polynomials. Consider the case when $U = \{0\}$ and $S = V$ and conclude that there exists a monic polynomial q_T such that for any polynomial f, we have $f(T) = 0$ iff $q_T | f$. The polynomial q_T is called the **minimal polynomial** of T.

Problem 5. Let $T : V \longrightarrow V$ be a linear transformation and let $v \in V$ be a non-zero vector.

a) Prove that any $T-$invariant subspace of a cyclic subspace is cyclic.

Hint. Let U be a T-invariant subspace of $< v >$. Consider the unique monic polynomial q

with the property that for any polynomial f, we have $f(T)(v) \in U$ iff $q|f$ (we proved in class that it exists). Show that $U =< w >$, where $w = q(T)(v)$. Show also that $q|p_v$.

b) Prove that if p_v is a power of an irreducible polynomial and U, W are T-invariant subspaces of $< v >$ then either $U \subseteq W$ or $W \subseteq U$. Conclude that $< v >$ cannot be decomposed into a direct sum of proper T-invariant subspaces.

c) Prove that a cyclic subspace $< v >$ cannot be decomposed into a direct sum of its proper $T-$invariant subspaces iff p_v is a power of an irreducible monic polynomial.
Hint. If p_v is not a power of an irreducible polynomial, then there are an irreducible polynomial q, a positive integer l and a polynomial f not divisible by q such that $p_v = q^l f$. Set $u = g^l(T)(v)$ and $w = f(T)(v)$ and show that $< v >=< u > \oplus < w >$.

d) Show that $< w >=< v >$ iff $w = f(T)(v)$ for some polynomial f relatively prime to p_v.

e) Let f be a polynomial. Describe p_w for $w = f(T)(v)$.
Hint. Show that $p_w = p_v/h$, where h is the greatest common divisor of p_v and f.