Homework 7

due on Wednesday, November 24

Problem 1. a) Let $f, g, h \in K[x]$ be polynomials. Suppose that $f|gh$ and $\gcd(f, g) = 1$. Prove that $f|h$ (modify the proof of Theorem 3 in the notes or use Problem 3 from Homework 6.).

b) Let $f_1(x), f_2(x), ..., f_k(x) \in K[x]$ be non-zero polynomials. Prove that there exists unique monic polynomial $m(x)$ such that

i) each polynomial $f_i(x)$ divides $m(x)$;

ii) if each $f_i(x)$ divides a polynomial $h(x)$ then $m(x)$ divides $h(x)$.

The polynomial m is called the **least common multiple** of $f_1, f_2, ..., f_k$ and it is denoted by $\text{lcm}(f_1, f_2, ..., f_k)$.

c) Prove that $\text{lcm}(f, g) = fg/\gcd(f, g)$.

Problem 2. Let $T : V \longrightarrow V$ be a linear transformation.

a) Let $v, w \in V$ be such that p_v and p_w are relatively prime (i.e. $\gcd(p_v, p_w) = 1$). Prove that $p_{v+w} = p_v p_w$. Show also that $p_{cv} = p_v$ for any non-zero constant c.

b) The result in a) may suggest that, in general, p_{v+w} is the lest common multiple of p_v and p_w. Show by example that this is false. Prove that p_{v+w} divides the polynomial $p_v p_w/\gcd(p_v, p_w)$ (which is the lest common multiple of p_v and p_w) and that the polynomial $p_v p_w/\gcd(p_v, p_w)^2$ divides p_{v+w}.

Problem 3. Let $a_0, a_1, ..., a_{n-1} \in K$. Prove that the determinant of the matrix

$$\begin{pmatrix} x & 0 & \cdots & 0 & 0 & a_0 \\ -1 & x & \cdots & 0 & 0 & a_1 \\ 0 & -1 & \cdots & 0 & 0 & a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & x & a_{n-2} \\ 0 & 0 & \cdots & 0 & -1 & x + a_{n-1} \end{pmatrix}$$

equals $x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$.

Problem 4. Find the minimal and characteristic polynomials of the linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ given by the matrix

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -3 & 2 & 2 & 0 & -2 \\ 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & -2 & 2 & 2 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Compute $M(p, k)$ for every irreducible polynomial p and every integer k.

Problem 5. Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ be given by $T(a, b, c, d) = (a + b, b + c, c + d, d + a)$.

a) Find the annihilator of $v = (1, 0, -1, 0)$ and of $w = (1, 0, 0, 0)$.

b) Find the minimal polynomial of T.

c) Find a rational canonical form of T and a basis in which T has this form.

Problem 6. Let $T : V \longrightarrow V$ be a linear transformation and let $v_1, ..., v_n$ be a basis of V. Prove that the minimal polynomial q_T is equal to the least common multiple of the annihilators $p_{v_1}, ..., p_{v_n}$ of $v_1, ..., v_n$.

Problem 7. Let $T \in L(V)$. Define a liner transformation $M_T : L(V) \longrightarrow L(V)$ by $M_T(S) = TS$.

a) Prove that the annihilator of the identity $I \in L(V)$ (with respect to M_T) is equal to the minimal polynomial of T.

b) Prove that p_S divides the minimal polynomial of T for any $S \in L(V)$. Conclude that the minimal polynomial of M_T coincides with the minimal polynomial of T.

c) Prove that the characteristic polynomial of M_T is the n-th power of the characteristic polynomial of T, where $n = \dim V$.