Homework 8
due on Monday, December 6

Problem 1. a) Let A be a Jordan block, i.e. an $n \times n$ matrix of the form

$$A = \begin{pmatrix} x & 0 & \cdots & 0 & 0 & 0 \\ 1 & x & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x & 0 \\ 0 & 0 & \cdots & 0 & 1 & x \end{pmatrix}.$$

Let f(x) be a polynomial. Prove that $f(A) = (a_{i,j})$, where $a_{i,j} = 0$ for $i < j$ and $a_{i,j} = f^{(i-j)}(x)/(i-j)!$ for $i \geq j$ (here $f^{(m)}$ denotes the m-th derivative of f), i.e.

$$f(A) = \begin{pmatrix} f(x) & 0 & \cdots & 0 & 0 & 0 \\ f'(x)/1! & f(x) & \cdots & 0 & 0 & 0 \\ f''(x)/2! & f'(x)/1! & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f^{(n-2)}(x)/(n-2)! & f^{(n-3)}(x)/(n-3)! & \cdots & f'(x)/1! & f(x) & 0 \\ f^{(n-1)}(x)/(n-1)! & f^{(n-2)}(x)/(n-2)! & \cdots & f''(x)/2! & f'(x)/1! & f(x) \end{pmatrix}.$$

In particular,

$$A^k = \begin{pmatrix} x^k & 0 & \cdots & 0 & 0 & 0 \\ \binom{k}{1}x^{k-1} & x^k & \cdots & 0 & 0 & 0 \\ \binom{k}{2}x^{k-2} & \binom{k}{1}x^{k-1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{k}{n-2}x^{k-n+2} & \binom{k}{n-3}x^{k-n+3} & \cdots & \binom{k}{1}x^{k-1} & x^k & 0 \\ \binom{k}{n-1}x^{k-n+1} & \binom{k}{n-2}x^{k-n+2} & \cdots & \binom{k}{2}x^{k-2} & \binom{k}{1}x^{k-1} & x^k \end{pmatrix}.$$

Here $\binom{a}{m} = a(a-1)...(a-m+1)/m!$.

Since every matrix over an algebraically closed field is similar to its Jordan canonical form (which is block-diagonal with Jordan blocks on the diagonal), the above result allows to "understand" powers of a given matrix (via the Jordan canonical form).

b) Suppose that M is an $n \times n$ matrix over the complex numbers. Prove that the infinite sum

$$I + M + M^2/2! + M^3/3! + ... = \sum_{k=0}^{\infty} M^k/k!$$

converges. The sum of this series is denoted by e^M. Prove that $e^{CMC^{-1}} = Ce^MC^{-1}$ for every invertible matrix C.

c) Prove that if $MN = NM$ then $e^{M+N} = e^M e^N$. Show that this is not true without the assumption that $MN = NM$.

d) Let A be a Jordan block as in a) and let t be a scalar. Prove that $e^{tA} = (a_{i,j})$, where $a_{i,j} = 0$ for $i < j$ and $a_{i,j} = t^{i-j}e^{tx}/(i-j)!$ for $i \geq j$, i.e.

$$e^{tA} = e^{tx}\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ t/1! & 1 & \cdots & 0 & 0 & 0 \\ t^2/2! & t/1! & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t^{n-2}/(n-2)! & t^{n-3}/(n-3)! & \cdots & t/1! & 1 & 0 \\ t^{n-1}/(n-1)! & t^{n-2}/(n-2)! & \cdots & t^2/2! & t/1! & 1 \end{pmatrix}.$$

This result and b) tell us how to compute e^M for any matrix M.

e) Prove that $\det(e^M) = e^{\mathrm{tr}M}$, where $\mathrm{tr}A$ is the trace of A. In particular, e^M is always invertible. What is its inverse?

Problem 2. In applications it is often necessary to consider matrices of the form $M(t) = (f_{i,j}(t))$, where each entry $f_{i,j}(t)$ is a function of t. If all the entries are differentiable we define $M'(t) = (f'_{i,j}(t))$.

a) Prove that $(A(t)B(t))' = A'(t)B(t) + A(t)B'(t)$ and $(A(t)+B(t))' = A'(t) + B'(t)$.

b) Prove that $(e^{tA})' = Ae^{tA}$ for any matrix A. In other words, if $c_i(t)$ is the i-th column of e^{tA} then $c_i'(t) = Ac_i(t)$.

In the theory of differential equations one often needs to consider the following problem. Given an $n \times n$ matrix $A = (a_{i,j})$ find functions $f_1(t)$, $f_2(t)$,...,$f_n(t)$ such that $f_i'(t) = \sum_{j=1}^n a_{i,j}f_j(t)$. If we set $\mathbf{f}$ for the column vector $(f_1(t), f_2(t), ..., f_n(t))^T$ (the T indicates transposition) then the problem takes form $\mathbf{f}'(t) = A\mathbf{f}(t)$. This is called a **homogeneous system of linear differential equations of order one with constant coefficients**. We may consider the solutions $\mathbf{f}$ as elements of the vector space of all the n-tuples of functions.

c) Prove that the set of all solutions to the above system is a subspace.

A theorem in the theory of differential equations states that given t_0 and numbers a_1, a_2,...,a_n there is unique solution $(f_1(t), f_2(t), ..., f_n(t))$ such that $f_i(t_0) = a_i$. In the language of linear algebra this simply means that the space of solutions has dimension n (why?). In b) we have seen that the columns of the matrix e^{tA} are solutions to the above system.

d) Prove that the columns of the matrix e^{tA} for a basis of solutions of the system $\mathbf{f}'(t) = A\mathbf{f}(t)$. More precisely, show that the unique solution such that $f_i(t_0) = a_i$ is given by the formula

$$\mathbf{f}(t) = e^{(t-t_0)A}\mathbf{a},$$

where $\mathbf{a} = (a_1, ..., a_n)^T$.

Another problem often considered in the theory of differential equations is to find all functions $f(t)$ such that $f^{(n)}(t) + a_{n-1}f^{(n-1)}(t) + ... + a_1f'(t) + a_0f(t) = 0$. This is called a **homogeneous linear differential equation of order n with constant coefficients**. The following observation reduces this problem to the previous one. To solve this equation is equivalent to solving the system

$$\begin{aligned} f_1'(t) &= f_2(t) \\ f_2'(t) &= f_3(t) \\ \vdots \quad &\vdots \quad \vdots \quad \vdots \\ f_{n-1}'(t) &= f_n(t) \\ f_n'(t) &= -a_{n-1}f_n(t) - \dots - a_1f_2(t) - a_0f_1(t) \end{aligned}$$

Here $f = f_1$. In particular, given t_0 and $a_1, \dots, a_n$ there is unique solution f such that $f^{(k)}(t_0) = a_{k+1}$, $k = 0, 1, \dots, n-1$.

e) Find the function f such that $f(0) = -1$, $f'(0) = 1$, $f''(0) = 6$, $f'''(0) = 10$ and $f''''(t) - 4f'''(t) + 8f''(t) - 8f'(t) + 4f(t) = 0$. Hint: Find the gcd of $x^4 - 4x^3 + 8x^2 - 8x + 4$ and its derivative (this should help you decompose the polynomial). Work over complex numbers (to find Jordan canonical forms).

Problem 3. Find a rational canonical basis and rational canonical form of the linear transformation $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ given by the matrix

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ -3 & 2 & 2 & 0 & -2 \\ 1 & 0 & 0 & 0 & 2 \\ 3 & 0 & -2 & 2 & 2 \\ 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Compute B^{1000}.

Problem 4. a) Find the minimal polynomial and eigenvalues of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Can you find a Jordan canonical form of this matrix without any further calculations?

b) Find Jordan canonical basis for the linear transformation whose matrix representation in the standard basis of $\mathbb{R}^3$ is

$$C = \begin{pmatrix} -1 & -2 & 0 \\ 2 & 3 & 0 \\ 3 & 3 & 1 \end{pmatrix}$$

knowing that the only eigenvalue of this matrix is 1. Find Q such that $Q^{-1}CQ$ is a Jordan canonical form of C.

Problem 5. Let $T : V \longrightarrow V$ be a linear transformation and let $V = < v_1 > \oplus \dots \oplus < v_l >$ be a rational canonical decomposition of V with respect to T.

a) Show that the annihilator of $v = v_1 + \dots + v_l$ is the minimal polynomial q_T of T.

Assume that the degree of q_T equals $n = dimV$. Then

b) There is a vector w such that $V = < w >$ (use a)).

c) Suppose that $S : V \longrightarrow V$ is another linear transformation. Since $w, T(w), ..., T^{n-1}(w)$ is a basis of $< w >= V$, there exist scalars $b_0, ..., b_{n-1}$ such that $S(w) = b_0 w + b_1 T(w) + ... + b_{n-1}T^{n-1}(w)$. Prove that if $ST = TS$ then $S = b_0 I + b_1 T + ... + b_{n-1}T^{n-1}$.
The moral of this problem is that if q_T has degree n then S commutes with T iff it is a polynomial in T.

Problem 6. a) Find an orthonormal basis of the subspace of $\mathbb{R}^5$ given by the equations $x_1 - x_2 + x_3 - x_4 = 0$, $2x_1 - x_3 - x_4 + x_5 = 0$. Find a basis of the orthogonal complement to this subspace.

b) Find the orthogonal projection of $v = (1, 0, 0, 0)$ onto the subspace W of $\mathbb{R}^4$ spanned by $(1, -1, -1, 1), (1, 1, -1, -1), (1, 1, 1, 1)$. What is the distance from v to W?

c) Let $v_1, ..., v_s$ be an orthonormal subset of an inner product space V. Show that $||v||^2 \geq |(v, v_1)|^2 + |(v, v_2)|^2 + ... + |(v, v_s)|^2$ for any vector v. Prove that the equality holds iff v is a linear combination of the vectors $v_1, ..., v_s$.

Problem 7. Let $T : V \longrightarrow V$ be a linear transformation of a finite dimensional inner product space over the field K such that $< T(v), v >= 0$ for all $v \in V$.

a) Show that if $K = \mathbb{C}$ then $T = 0$.

b) Prove that if $K = \mathbb{R}$ and T is self-adjoint then $T = 0$.

c) Prove that if $K = \mathbb{R}$ then satisfies the assumptions iff $T^* = -T$.

Problem 8. Let $T : V \longrightarrow V$ be a self-adjoint linear transformation of a finite dimensional inner product space such that $T^2 = T$. Prove that $T = P_W$ is the orthogonal projection onto some subspace W.

Problem 9. Let $T : V \longrightarrow V$ be a linear transformation of a finite dimensional inner product space.

a) Prove that $\mathrm{Im}(T^*) = (\ker T)^{\perp}$ and $\ker T^* = \mathrm{Im}(T)^{\perp}$.

b) Prove that if T is normal then $\ker T^* = \ker T$ and $\mathrm{Im}(T^*) = \mathrm{Im}(T)$

c) Prove that if T is normal and the field of scalars is $\mathbb{C}$ then T has a square root, i.e. there is a linear transformation $S : V \longrightarrow V$ such that $S^2 = T$.

d) Prove that if T is self-adjoint and $< Tv, v >$ is a non-negative real number for all $v \in V$ (such T are called **positive**) then there is unique positive square root of T. Conclude that $T = SS^*$ for some $S : V \longrightarrow V$.

Problem 10. Let $T : V \longrightarrow V$ be a linear transformation of a finite dimensional inner product space. Prove that the following conditions are equivalent:

1. T is normal;
2. $\| T(v) \| = \| T^*(v) \|$ for all $v \in V$;
3. $\| T(v) \| \geq \| T^*(v) \|$ for all $v \in V$;
4. $\| T(v) \| \leq \| T^*(v) \|$ for all $v \in V$.